

FERMI ACCELERATION IN ANTI-INTEGRABLE LIMITS OF THE STANDARD MAP

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ABSTRACT. We consider a dynamical system on the semi-infinite cylinder which models the high energy dynamics of a family of mechanical models. We provide conditions under which we ensure that the set of orbits undergoing Fermi acceleration has measure zero.

1. INTRODUCTION AND RESULTS

In this work we study dynamical properties of a family of exact area-preserving twist maps on the semi-infinite cylinder. This family describes an approximation of the high energy dynamics of a class of generalizations [11, 12, 9, 10, 3] of the Fermi-Ulam ping-pong [14]. Among other examples, the dynamics of some n -body problems, such as the Sitnikov three-body problem (see [13, 7]) can also be described by means of maps which are similar to the ones we consider in this work.

One of the most remarkable differences between finite and infinite measure dynamical systems is that the latter are possibly lacking the *recurrence* property, which is guaranteed by Poincaré theorem in the former situation. If the set of *wandering* (that is, non-recurrent) points has positive measure, we say that the system is *dissipative*, otherwise we say it is *conservative*. Conservativity is a very desirable property for infinite measure systems and it is the starting point to prove (and even to define in a satisfactory manner) ergodic and statistical properties such as bounds for the decay of correlations (see for instance [8]). In our framework, conservativity has also a very concrete physical interpretation for the mechanical systems which our maps relate to; in our case, in fact, if a point belongs to the wandering set, then, necessarily, its energy will tend to infinity with time; they are in fact orbits which undergo so-called *Fermi acceleration* (see [5, 6]). In this paper we show that, if suitable conditions of the parameters are satisfied, the maps of our family are conservative; consequently, the mechanical systems which can be modeled by the above choice of parameters, allow only a null set of Fermi accelerated orbits.

Another quite interesting feature of the family under consideration is its affinity with the Chirikov-Taylor standard map (see [1]). For instance (see e.g. [11]) the Fermi-Ulam ping-pong system is well described, for large values of the non-cyclic variable (i.e. for high energies), by the dynamics of the standard map for small coupling parameter, i.e. in the quasi-integrable regime. On the other hand, the family of maps under our consideration are such that, for large values of the non-cyclic variable, the dynamics is described by the standard map for large coupling parameters, that is, far away from the integrable regime. In this sense our maps can be regarded

as anti-integrable limits of the standard map. For this reason, we believe that our family shares part of the “universality” properties of the standard map and that this study can indeed be useful for a variety of different situations. Additionally, most of the difficulties we will encounter in our work will be directly related to corresponding issues for the standard map and we can expect the techniques employed in the present work to be successfully applied also to that more challenging study.

Let $\mathbb{T}^1 \doteq \mathbb{R}/2\pi\mathbb{Z}$ and $\mathbb{A} \doteq \mathbb{T}^1 \times \mathbb{R}_+$ be the semi-infinite cylinder. Let ϕ be a smooth real-valued function on \mathbb{T}^1 ; for definiteness we assume

$$(1.1) \quad \phi(\theta) = \sin(\theta),$$

and for $A > 0$ we let $\phi_A \doteq A\phi$. For $\hat{Y}, \gamma \in (0, \infty)$ fix Y_γ to be a smooth orientation-preserving diffeomorphism of \mathbb{R}_+ given by:

$$(1.2) \quad Y_\gamma(y) = \hat{Y} \cdot y^\gamma.$$

Finally, let $F_{A,\gamma} : \mathbb{A} \rightarrow \mathbb{A}$ be the area-preserving map given by the following formula:

$$(1.3) \quad F_{A,\gamma} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + Y_\gamma(y) \\ y + 2\dot{\phi}_A(x + Y_\gamma(y)) \end{pmatrix},$$

if $y > L$ for some L large enough, and continued to a smooth map on $0 \leq y \leq L$; the exact form of the continuation is irrelevant for our statements, since it only influences the dynamics on a compact region of the phase space. We can always assume $\hat{Y} = 1$, otherwise we let $y \mapsto \hat{Y}^{1/\gamma}y$ and $A \mapsto \hat{Y}^{1/\gamma}A$. We fix once and for all the values of A and γ and then study the dynamics of $F_{A,\gamma}$; for sake of simplicity we will therefore drop all subscripts A and γ appearing in the definitions since this will not be source of confusion. Furthermore, introduce the convenient notation $(x_k, y_k) = F^k(x_0, y_0)$; we define the *escaping set* \mathcal{E} as follows:

$$\mathcal{E} \doteq \{(x_0, y_0) \text{ s.t. } \lim_{n \rightarrow \infty} y_n = \infty\}.$$

In our setting the escaping set coincides with the wandering set; however, since we find the word “escaping” more descriptive, we prefer it over our other option. Our interest is to provide results on the largeness of the set \mathcal{E} depending on the values of A and γ .

The map given by (1.3) can be obtained (see [2] for a detailed derivation) as an high energy approximation (the so called *static wall approximation*) of a suitable Poincaré map of the dynamics of a particle bouncing on a periodically oscillating infinitely heavy plate while subject to a potential force given by a power law with exponent $2/(\gamma + 1)$. In this model the x variable corresponds to the time of a collision with the moving plate, while y corresponds to the post-collisional velocity; some results, which have been originally obtained for the mechanical problem, can indeed be adapted to our situation; we list a selection of them, which are directly related to this work.

Theorem 1.1 (Pustynnikov [11]). *If $\gamma = 1$, then the set of escaping orbits contains an open set for an open set of values of A .*

Theorem 1.2 (Ortega [9]). *If $\gamma = 0$ and certain resonance conditions are satisfied, then the set of escaping orbits \mathcal{E} contains an open set of the phase space.*

Theorem 1.3 (Dolgopyat [3]). *If $\gamma \in (0, 1)$, then the set of escaping orbits \mathcal{E} is empty.*

The proof of Theorem 1.1 is indeed quite simple in our case; in fact, if $\gamma = 1$, then the map F for large y is given by the unfolding of the standard map on the cylinder. In this case it is easy to prove that, for an open set of parameters, we can find a periodic orbit on \mathbb{T}^2 given by centers of a chain of elliptic islands of period N for the standard map on the torus \mathbb{T}^2 which lifts to a non-periodic orbit on the cylinder such that $F^N : (x, y) \mapsto (x, y + \nu)$, where ν is a positive integer. This implies that the lift of any elliptic island in the chain is a subset of the escaping set, which consequently has positive, hence infinite, measure. The statement of Theorem 1.2 is also not surprising, in fact if $\gamma = 0$, then the function Y is constant; then if we have a resonance condition between Y and the period of ϕ , it is plausible that escaping orbits can indeed arise. Finally, Theorem 1.3 follows from showing the existence of KAM tori for large values of y ; in fact their presence prevents diffusion and hence implies the result. On the one hand, for values of γ greater than 1, the set of escaping orbits \mathcal{E} is non-empty; in fact it was proved in [2] that the set has full Hausdorff dimension. On the other hand, the following result holds:

Theorem 1.4 (Dolgopyat [3]). *If $\gamma > 5$, then the set of escaping orbits \mathcal{E} has zero measure.*

In the same article it is indeed conjectured that

Conjecture 1.5. *If $\gamma > 1$, then the set of escaping orbits \mathcal{E} has zero measure.*

The main result of this work is an improvement of Theorem 1.4 and constitutes a step towards the proof of the conjecture.

Main Theorem. *If $\gamma > 2$, then the set of escaping orbits \mathcal{E} has zero measure.*

As it will be clear once the proof will be explained, our strategy for the proof does not work if $\gamma \leq 2$. By performing some numerical simulations, it seems likely that this is not merely a technical problem; indeed, proving our Main Theorem for the value $\gamma = 2$ seems to require a somewhat different approach, and it is still out of reach at the moment.

Additionally, the techniques developed in order to prove our Main Theorem are of independent interest for the study of statistical properties of non-uniformly hyperbolic systems which appear to present coexistence of elliptic and hyperbolic behavior; as will be made clear later, one can regard the condition on γ as a stipulation on how fast the expansion rate of the map along the cyclic coordinate x can grow along with the non-cyclic coordinate y : if the growth rate is strong enough, then we can conclude that the system is conservative. Indeed, a part from the classical example of the standard map, we believe that our techniques could be employed in more general analyses (e.g. the one provided in [4]).

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2. PROOF OF OUR MAIN THEOREM

The set \mathcal{E} is invariant, and thus, by Poincaré recurrence Theorem, has either zero or infinite measure. Additionally, if $(x, y) \in \mathcal{E}$, then for any $y_* > 0$ there exists a k_* such that if $k > k_*$, $y_k \geq y_*$, therefore if we let

$$\mathcal{E}_* = \{(x_0, y_0) \in \mathcal{E} \text{ s.t. } y_k \geq y_* \text{ for } k \geq 0\}$$

we conclude that $\mathcal{E} = \bigcup_{k \geq 0} F^{-k} \mathcal{E}_*$, thus, in order to prove our Main Theorem, it suffices to prove that there exists a $y_* > 0$ such that \mathcal{E}_* has zero measure. It is convenient to introduce the notation $\mathbb{A}_* = \{y \geq y_*\}$; we will need to choose y_* very large in order to satisfy a number of requirements, which will be stated in due course; in particular we will always assume $y_* > L$ where L is the constant introduced in the previous section.

The proof is based on the idea used for the proof of Theorem 1.4, however some substantial improvements are required. The proof of Theorem 1.4 is based on equidistribution on \mathbb{T}^1 of x -components of almost all trajectories which do not leave \mathbb{A}_* in the future; this ultimately follows from proving that the expansion along so-called *standard curves* provides enough uniformity to deliver equidistribution outside of a *critical set* \mathbf{C} which has finite measure for large enough γ . On the other hand, on \mathbf{C} the dynamics is recurrent by Poincaré's Theorem and, therefore, orbits which return to \mathbf{C} infinitely many times cannot escape. Equidistribution on the complement of \mathbf{C} allows then to set up a comparison with a random walk which ultimately is used to prove that almost every trajectory will eventually land in \mathbf{C} . By sharpening some estimates, we can push this strategy to work up to $\gamma > 3$ but it ultimately fails for any smaller γ , since the measure of the critical set will necessarily be infinite in this case. To obtain the result for $\gamma > 2$, we do need to study the dynamics of the map inside the critical set. In particular, the main strategy to improve the condition on γ is to recover some hyperbolicity of the dynamics inside \mathbf{C} by considering successive iterations of the map; by doing so we obtain a smaller critical set $\mathbf{C}_* \subset \mathbf{C}$ whose measure will be finite also for smaller values of γ . In [2] we proved that the measure of elliptic islands in the critical set is infinite if $\gamma \leq 4/3$; this implies that the above strategy cannot be used to fully prove Conjecture 1.5. The present work shows, however, that the strategy can be successfully employed to prove our Main Theorem, for which it is sufficient to consider a critical set obtained with a two-iterate scheme (critical set of order 2); however, in order to obtain the necessary equidistribution estimates, one needs to consider several iterates; the number of iterates in fact tends to ∞ as $\gamma \rightarrow 2$. Thus, in a sense, our strategy is optimal when using critical sets of order 2.

2.1. Standard pairs. We will study equidistribution properties of the dynamics employing the technique of *standard pairs*. A curve $\Gamma \subset \mathbb{A}_*$ is said to be a *basic curve* if it is a graph of a smooth function $\psi : I \rightarrow \mathbb{R}$ where $I \subset \mathbb{T}^1$ is an interval. A *basic pair* ℓ is then given by a basic curve Γ_ℓ and a strictly positive smooth probability density ρ_ℓ on I : we write $\ell = (\Gamma_\ell, \rho_\ell)$ where $\Gamma_\ell = (x, \psi_\ell(x))$ for $x \in I_\ell$. A basic pair defines a measure as follows: for any real valued Borel measurable function $\mathcal{A}(x, y)$ we define:

$$\mathbb{E}_\ell(\mathcal{A}) \doteq \int_{\Gamma_\ell} \mathcal{A} \rho_\ell dx = \int_{I_\ell} \mathcal{A}(x, \psi_\ell(x)) \rho_\ell(x) dx,$$

and for any Borel measurable set E :

$$\mathbb{P}_\ell(E) \doteq \mathbb{E}_\ell(1_E).$$

We introduce, for convenience, the function $Y_\ell(x) = Y(\psi_\ell(x))$ and similarly $Y'_\ell(x) = Y'(\psi_\ell(x))$ and $Y''_\ell = Y''(\psi_\ell(x))$. We denote by h_ℓ the *slope* of the basic curve Γ_ℓ , i.e. $h_\ell(x) \doteq \dot{\psi}_\ell(x)$, where the dot denotes differentiation with respect to the variable x . It is also useful to introduce the *adapted slope* function $\tilde{h}_\ell(x) \doteq h_\ell(x) + 1/Y'_\ell(x)$ and the *local expansion rate* $\mathcal{L}_\ell(x) = \tilde{h}_\ell(x)Y'_\ell(x)$. Notice that definition (1.3) implies that, if $(x, y) \in \Gamma_\ell$ and $(x', y') = F(x, y)$:

$$\mathcal{L}_\ell(x) = \left. \frac{dx'}{dx} \right|_{\Gamma_\ell} (x).$$

We denote by $r_\ell(x) \doteq \rho_\ell^{-1}(x)\dot{\rho}_\ell(x)$ the logarithmic derivative of ρ_ℓ . Finally, we define:

$$\hat{y}_\ell \doteq \inf_{x \in I} \psi_\ell(x).$$

Lemma 2.1. *Let $x_* \in I_\ell$ such that $\tilde{h}_\ell(x_*) \neq 0$; then there exists $U \subset I_\ell$ a neighborhood of x_* such that:*

- *the curve $\Gamma' = F\Gamma|_U$ is the graph of a smooth function $\psi' : I' \rightarrow \mathbb{R}$;*
- *the pushforward $\rho'(x') = c'\rho(x(x'))/\mathcal{L}_\ell(x(x'))$, where $x(x') = \pi_1 F^{-1}\Gamma'(x')$ and c' is a normalizing constant, is a strictly positive smooth probability density on I' ;*

hence $\ell' = (\Gamma', \psi')$ is a basic pair. Moreover:

$$(2.1a) \quad h_{\ell'} = 2\ddot{\phi}(x') + \frac{1}{Y'_\ell} \left(1 - \frac{1}{\mathcal{L}_\ell} \right)$$

$$(2.1b) \quad \dot{h}_{\ell'} = 2\ddot{\phi}(x') + \frac{\dot{h}_\ell}{\mathcal{L}_\ell^3} - \frac{Y''_\ell}{Y_\ell'^3} \left(1 - \frac{1}{\mathcal{L}_\ell} \right)^3$$

$$(2.1c) \quad r_{\ell'} = \frac{r_\ell}{\mathcal{L}_\ell} - \frac{\dot{h}_\ell Y'_\ell}{\mathcal{L}_\ell^2} - \frac{Y''_\ell}{Y_\ell'^2} \left(1 - \frac{1}{\mathcal{L}_\ell} \right)^2,$$

where all functions with subscript ℓ are evaluated at the point x and all functions with subscript ℓ' are evaluated at the corresponding point x' .

Proof. Equations (2.1) immediately follow from the definitions assuming $\mathcal{L}_\ell \neq 0$; on the other hand, if $\tilde{h}_\ell(x_*) \neq 0$ we know that there necessarily exists a neighborhood U such that $\tilde{h}_\ell(U) \not\ni 0$. Therefore, (2.1b) implies that the curve Γ' is a graph of a smooth function and $\mathcal{L}_\ell \neq 0$ implies that ρ' is strictly positive.

Notice moreover that even if ρ' depends on the choice of U , equations (2.1) are well-defined and independent of U . \square

We will shortly introduce the notion of *standard pairs*, which are given by a special class of basic pairs. First, define a class of basic pairs that we call *reference pairs*: geometrically, reference pairs are given by pieces of the image of a vertical line which are not too short nor too long endowed with a uniform density. Standard pairs will in turn be defined as being appropriately close to reference pairs.

Fix once and for all a sufficiently small $\delta > 0$; we require δ to be smaller than the minimum distance between two consecutive critical points of $\dot{\phi}$; in our case it suffices to take $\delta < \pi/4$. We say that an interval $I \subset \mathbb{T}^1$ is a *standard interval* if $\delta/4 < |I| < \delta$.

Definition 2.2. A basic curve $\Gamma = (x, \psi(x))$ with $\psi : I \rightarrow \mathbb{R}$ is said to be a reference curve if I is a standard interval and

$$\psi_\ell(x) = 2\dot{\phi}(x) + Y^{-1}(c + x)$$

for some $c > 0$; a basic pair ℓ is said to be a reference pair if Γ_ℓ is a reference curve and $\rho_\ell \equiv |I_\ell|^{-1}$.

Define the following functions:

$$(2.2) \quad h_1(x_0, y_0) = 2\ddot{\phi}(x_0) + \frac{1}{Y'(y_{-1})} \quad \dot{h}_1(x_0, y_0) = 2\ddot{\phi}(x_0) - \frac{Y''(y_{-1})}{Y'^3(y_{-1})}.$$

Then if ℓ is a reference pair, we have:

$$h_\ell(x) = h_1(x, \psi_\ell(x)) \quad \dot{h}_\ell(x) = \dot{h}_1(x, \psi_\ell(x)).$$

It is also convenient to define the function $\tilde{h}_1(x, y) = h_1(x, y) + 1/Y'(y)$; we will always require y_* to be so large that $\|h_1\|_{\mathbb{A}_*} < 3A$ and $\|\tilde{h}_1\|_{\mathbb{A}_*} < 3A$.

Definition 2.3. Let $I \subset \mathbb{T}^1$ be an interval and ρ a probability density on I ; we say that ρ is regular if $r(x) = \rho^{-1}(x)\dot{\rho}(x)$ satisfies $\|r\|_I < 1$.

Lemma 2.4. Let I be a standard interval; then there exist $0 < \mu_1 < \mu_2$ such that if ρ is a regular probability density on I , then for any measurable set $E \subset I$ we have

$$\mu_1 \text{Leb}(E) < \mathbb{P}(E) < \mu_2 \text{Leb}(E)$$

Proof. By the regularity condition $|r| < 1$, applying Grönwall lemma to ρ we obtain, for every $x, \bar{x} \in I$:

$$\rho(\bar{x})e^{-|x-\bar{x}|} \leq \rho(x) \leq \rho(\bar{x})e^{|x-\bar{x}|},$$

by taking \bar{x} such that $\rho(\bar{x}) = \bar{\rho}$ the average density and by the definition of standard interval we obtain:

$$\mu_1 = \delta^{-1}e^{-\delta} < \rho(x) < 4\delta^{-1}e^{\delta} = \mu_2 \quad \square$$

Definition 2.5. Fix D a constant to be defined later; let ℓ be a basic pair and define $\Delta h_\ell \doteq h_\ell(x) - h_1(x, \psi_\ell(x))$ and correspondingly $\Delta \dot{h}_\ell \doteq \dot{h}_\ell(x) - \dot{h}_1(x, \psi_\ell(x))$. Then ℓ is said to be a standard pair if I_ℓ is a standard interval,

ρ_ℓ is regular and Γ_ℓ is locally close to a reference curve in the following sense:

$$(2.3a) \quad |\Delta h_\ell(x)| < D^{-1}Y'_\ell(x)^{-3/2}$$

$$(2.3b) \quad |\Delta \dot{h}_\ell(x)| < A/10$$

The next lemma ensures that standard curves are globally close to reference curves.

Lemma 2.6. *Let ℓ be a basic pair and $\bar{\ell}$ a reference pair such that $I_\ell = I_{\bar{\ell}} = I$; assume there exists a $x_* \in I$ such that $\psi_\ell(x_*) = \psi_{\bar{\ell}}(x_*)$. Then:*

$$\forall x \in I \quad |\psi_\ell(x) - \psi_{\bar{\ell}}(x)| < 2\|\Delta h_\ell\||x - x_*|.$$

Proof. Let $\hat{y} = \min\{\hat{y}_\ell, \hat{y}_{\bar{\ell}}\}$ and let

$$\mu = \sup_{x \in \mathbb{T}^1} \left| \frac{\partial h_1}{\partial y}(x, \hat{y}) \right|;$$

it is immediate to check that for all $x \in \mathbb{T}^1$, $y \geq \hat{y}$ we have $\left| \frac{\partial h_1}{\partial y}(x, y) \right| \leq \mu = o(\hat{y}^{-1})$; therefore we can write:

$$\begin{aligned} \left| \frac{d}{dx} (\psi_\ell(x) - \psi_{\bar{\ell}}(x)) \right| &\leq |h_\ell(x) - h_1(x, \psi_\ell(x))| + |h_1(x, \psi_\ell(x)) - h_1(x, \psi_{\bar{\ell}}(x))| \\ &\leq \|\Delta h_\ell\| + \mu |\psi_\ell(x) - \psi_{\bar{\ell}}(x)|. \end{aligned}$$

Let $J \subset I$ be the connected component of the set $\{|\psi_\ell(x) - \psi_{\bar{\ell}}(x)| < 2\|\Delta h_\ell\|\}$ containing x_* ; for all $x \in J$ and for large enough \hat{y} we have:

$$\left| \frac{d}{dx} (\psi_\ell(x) - \psi_{\bar{\ell}}(x)) \right| \leq (1 + 2\mu)\|\Delta h_\ell\| \leq 2\|\Delta h_\ell\|,$$

which in particular implies that $J = I$ and concludes the proof. \square

Fix K large and an interval $I \in \mathbb{T}^1$ and let $S_I \in \mathbb{A}$ be the half-strip given by $I \times [K, \infty)$. We define *adapted coordinates* on S by straightening the foliation of S_I given by reference curves. More precisely:

Definition 2.7. *Fix $\bar{x} \in I$ and let*

$$\begin{aligned} \kappa : \tilde{I} \times \mathbb{R}^+ &\rightarrow I \times \mathbb{R} \\ (\xi, \eta) &\mapsto (\xi + \bar{x}, \psi_\eta(\xi + \bar{x})) \end{aligned}$$

where ψ_η is a reference curve such that $\psi_\eta(\bar{x}) = \eta$. We define adapted coordinates on S_I by taking the restriction of κ on $\kappa^{-1}S_I$

2.2. Critical sets. We need to establish results regarding invariance properties of standard pairs; in order to do so we need to obtain good geometrical and regularity bounds (to control h , \dot{h} and r) for the map F . Such bounds cannot be established everywhere; points where this is not possible will belong to sets that we will call *critical sets*. The definition of the critical sets depends on our requirements for a “good” bound, and therefore it is far from being unique. However, all critical sets need to satisfy the following condition: every orbit that never visits the critical sets is hyperbolic.

Definition 2.8. Fix K_1, K_2 large; we define \mathbf{C}_1 the critical set of order 1 and \mathbf{C}_2 the critical set of order 2 as follows:

$$\begin{aligned}\mathbf{C}_1 &\doteq \left\{ (x_0, y_0) \in \mathbb{A}_* \text{ s.t. } |\tilde{h}_1(x_0, y_0)| < K_1 Y'(y_0)^{-1/2} \right\}; \\ \mathbf{C}_2 &\doteq \left\{ (x_0, y_0) \in \mathbb{A}_* \text{ s.t. } |\tilde{h}_1(x_0, y_0) \tilde{h}_1(x_1, y_1)| < K_2 Y'(y_0)^{-1} \right\} \cap \mathbf{C}_1;\end{aligned}$$

Take $\bar{K}_2 > 4$ and define the set:

$$\bar{\mathbf{C}}_2 \doteq \left\{ (x_0, y_0) \in \mathbb{A}_* \text{ s.t. } |\tilde{h}_1(x_0, y_0)| < \bar{K}_2 Y'(y_0)^{-1} \right\}.$$

We choose K_2 so large that $\bar{\mathbf{C}}_2 \subset \mathbf{C}_2$; the set $\bar{\mathbf{C}}_2$ will be called the *core* of the critical set \mathbf{C}_2 . We furthermore assume y_* to be large enough so that $\{\ddot{\phi}(x) = 0\} \subset \bar{\mathbf{C}}_2$. Notice moreover that:

$$Y'(y_k) = Y'(y_0) (1 + \mathcal{O}(|k|y_0^{-1}))$$

which yields, for any given k :

$$(2.4) \quad \forall \varepsilon > 0 \exists \bar{y} \text{ s.t. } y_0 > \bar{y} \Rightarrow (1 - \varepsilon)Y'(y_0) < Y'(y_k) < (1 + \varepsilon)Y'(y_0).$$

Thus we can choose K_1 large enough to ensure that $\mathbf{C}_1 \cap F^{-1}\mathbf{C}_1 \subset \mathbf{C}_2$. We now proceed to define the *augmented* critical sets, which are suitably defined neighborhood of the critical sets. Fix $\hat{K}_1 > K_1$ to be determined later and define the following set:

$$(2.5) \quad \hat{\mathbf{C}}_1 \doteq \{(x_0, y_0) \in \mathbb{A}_* \text{ s.t. } |\tilde{h}_1(x_0, y_0)| < \hat{K}_1 Y'(y_0)^{-1/2}\}.$$

We now extend \mathbf{C}_2 to $\hat{\mathbf{C}}_1$:

$$\mathbf{C}_2^* \doteq \left\{ (x_0, y_0) \in \mathbb{A}_* \text{ s.t. } |\tilde{h}_1(x_0, y_0) \tilde{h}_1(x_1, y_1)| < K_2 Y'(y_0)^{-1} \right\} \cap \hat{\mathbf{C}}_1;$$

we furthermore require K_2 to be so large that the inclusion $F^{-1}\mathbf{C}_1 \cap \hat{\mathbf{C}}_1 \subset \mathbf{C}_2^*$ holds. Then, fix $\hat{K}_2 > K_2$ also to be determined later and define:

$$\hat{\mathbf{C}}_2 \doteq \{(x, y) \in \mathbb{A}_* \text{ s.t. } |\tilde{h}_1(x_0, y_0) \tilde{h}_1(x_1, y_1)| < \hat{K}_2 Y'(y_0)^{-1}\} \cap \hat{\mathbf{C}}_1$$

We describe in a lemma the geometrical features of critical sets, which are sketched in Figure 1. The proof of the lemma will be given in appendix A; for clarity, let us first introduce the following natural notion: given a basic curve Γ and a point $(x, y) \in \Gamma$, for any $r > 0$, we let the Γ -ball of radius r around (x, y) be the set of points $(x', y') \in \Gamma$ such that $|x' - x| < r$; this induces the corresponding notion of Γ -neighborhood of a subset of Γ .

Lemma 2.9. *The critical sets enjoy the following properties:*

- (a₁) for fixed $\Delta_1 > 0$, we can choose \hat{K}_1 so large that, for any standard curve Γ , the Γ -neighborhood of radius $r = \Delta_1 \hat{y}_\Gamma^{-\beta}$ of $\Gamma \cap \mathbf{C}_1$ is contained in $\hat{\mathbf{C}}_1$;
- (a₂) for fixed $\Delta_2 > 0$, we can choose \hat{K}_2 so large that, for any standard curve Γ , the intersection of the Γ -neighborhood of radius $r = \Delta_2 \hat{y}_\Gamma^{-2\beta}$ of $\Gamma \cap \mathbf{C}_2^*$ with $\hat{\mathbf{C}}_1$ is contained in $\hat{\mathbf{C}}_2$.
- (b₁) for any standard pair ℓ we have $\mathbb{P}_\ell(\hat{\mathbf{C}}_1) = \mathcal{O}(\hat{y}_\ell^{-\beta})$;
- (b₂) for any standard curve Γ , the number of connected components of $\Gamma \cap \hat{\mathbf{C}}_2$ is bounded uniformly in \hat{y}_Γ .
- (c₁) the Lebesgue measure of $\hat{\mathbf{C}}_1$ is finite if $\gamma > 3$;

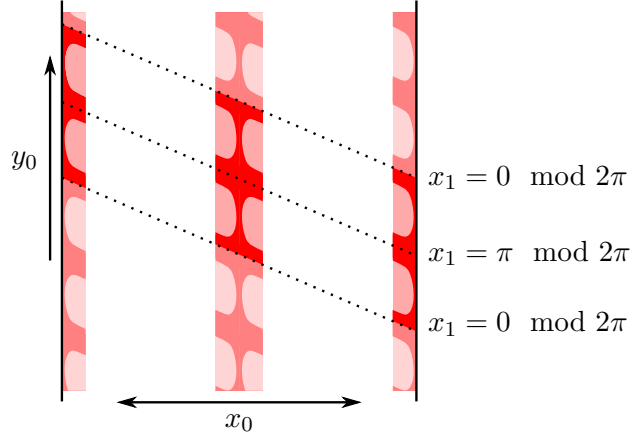


FIGURE 1. Sketch of the geometry of \mathbf{C}_1 and \mathbf{C}_2 for large y : \mathbf{C}_1 is given by the vertical strips in light color; \mathbf{C}_2 is given by the darker region inside \mathbf{C}_1 . We highlight a “fundamental domain” of \mathbf{C}_2 ; the reader can check that this is an accurate depiction of \mathbf{C}_1 and \mathbf{C}_2 by simple inspection of the definition.

(c_2) the Lebesgue measure of $\hat{\mathbf{C}}_2$ is finite if $\gamma > 2$;

As we mentioned at the beginning of this subsection, on critical sets we lack good geometrical and regularity estimates that can be achieved on the complementary set. In particular, outside \mathbf{C}_1 standard pairs will be mapped to standard pairs; pieces of standard pairs passing through the first critical set will possibly be mapped to non-standard pairs. However, pieces of standard pairs that lie in $\hat{\mathbf{C}}_1 \setminus \mathbf{C}_2^*$ are guaranteed to be standard after one more iteration. In the following lemma we prove the previous statements and establish some expansion bounds which will be crucial for proving equidistribution properties of F along the horizontal direction.

Definition 2.10. A standard partition \mathcal{J} of the circle \mathbb{T}^1 is a partition mod 1 in a finite number of closed intervals $\mathcal{J} = \{J_\alpha\}$, $\alpha \in \mathcal{A}$ satisfying the following conditions:

$$\text{int } J_\alpha \cap \text{int } J_{\alpha'} = \emptyset \text{ if } \alpha \neq \alpha', \quad \mathbb{T}^1 = \bigcup_{\alpha \in \mathcal{A}} J_\alpha, \quad \delta/4 < |J_\alpha| < \delta/2.$$

A basic pair ℓ is said to be \mathcal{J} -aligned if $I_\ell \in \mathcal{J}$.

Lemma 2.11 (Invariance). Fix a standard pair $\ell = (\Gamma_\ell, \rho_\ell)$ and a standard partition \mathcal{J} ; let $\hat{y} = \hat{y}_\ell$, $Y = Y(\hat{y})$ and similarly for Y' . Then we can choose y_* large enough so that:

(a) the following estimates hold:

$$(2.6a) \quad \left| \frac{dx_1}{dx_0} \right|_{\Gamma_\ell} > \frac{1}{2} K_1 Y'^{1/2} \quad \text{if } (x_0, y_0) \notin \mathbf{C}_1$$

$$(2.6b) \quad \left| \frac{dx_1}{dx_0} \right|_{\Gamma_\ell} > \frac{1}{2} \bar{K}_2 > 1 \quad \text{if } (x_0, y_0) \notin \mathbf{C}_2$$

$$(2.6c) \quad \left| \frac{dx_2}{dx_0} \right|_{\Gamma_\ell} > \frac{1}{2} K_2 Y' \quad \text{if } (x_0, y_0) \in \hat{\mathbf{C}}_1 \setminus \mathbf{C}_2^*$$

(b) we can uniquely decompose $F\ell$ as follows:

$$(2.7) \quad F\ell = \bigcup_{\alpha \in \mathcal{A}} \bigcup_j \ell_\alpha^j \cup \ell_+ \cup \ell_- \cup \bigcup_j \tilde{\ell}_j \cup Z$$

such that:

- each ℓ_α^j is a \mathcal{J} -aligned standard pair and $I_{\ell_\alpha^j} = J_\alpha$
- ℓ_+ and ℓ_- might be either empty or standard pairs such that $F^{-1}\ell_\pm \cap \mathbf{C}_1 = \emptyset$.
- each $\tilde{\ell}_j$, which we call a stand-by pair, is such that we have $F\tilde{\ell}_j = \bigcup_l \ell_{j,l}$ where $\ell_{j,l}$ are standard pairs;
- the number of stand-by pairs is bounded uniformly in y ;

Moreover:

$$(2.8) \quad \ell \cap \mathbf{C}_1 \subset F^{-1}\tilde{\ell}_j \subset \ell \cap \hat{\mathbf{C}}_1 \quad \ell \cap \mathbf{C}_2 \subset F^{-1}Z \subset \ell \cap \hat{\mathbf{C}}_2$$

Proof. Recall that by definition:

$$\left. \frac{dx_1}{dx_0} \right|_{\Gamma_\ell} (x) = \mathcal{L}_\ell(x) = \tilde{h}_\ell(x) Y'_\ell(x);$$

moreover, if $(x, y) \notin \mathbf{C}_1$ we have that $|\tilde{h}_1(x, y)| \geq K_1 Y'(y)^{-1/2}$, and if $(x, y) \notin \mathbf{C}_2$ we have that $|\tilde{h}_1(x, y)| \geq \bar{K}_2 Y'(y)^{-1}$. Then, by (2.3a) we immediately obtain (2.6a) and (2.6b) provided that y_* is large enough. Additionally, we can conclude that outside $\bar{\mathbf{C}}_2$ we have $h_\ell(x) \neq 0$, thus we can apply Lemma 2.1 and using (2.6b):

$$(2.9) \quad |h_{\ell'}(x_1) - h_1(x_1, y_1)| \leq 2(\bar{K}_2 Y')^{-1};$$

hence, since ℓ is standard we obtain the following bound if $(x_0, y_0) \in \hat{\mathbf{C}}_1 \setminus \mathbf{C}_2^*$:

$$|\tilde{h}_\ell(x_0) \tilde{h}_{\ell'}(x_1)| \geq 3/4 K_2 Y'^{-1}$$

which implies (2.6c) since

$$\left. \frac{dx_2}{dx_0} \right|_{\Gamma_\ell} = \tilde{h}_\ell(x_0) Y'_\ell(x_0) \tilde{h}_{\ell'}(x_1) Y'_{\ell'}(x_1).$$

In order to prove part (b) first of all notice that if $(x_0, y_0) \notin \mathbf{C}_1$ we can apply Lemma 2.1 and part (a) obtaining:

$$(2.10a) \quad |h_{\ell'}(x_1) - h_1(x_1, y_1)| \leq 2K_1^{-1} Y'^{-3/2}$$

$$(2.10b) \quad |\dot{h}_{\ell'}(x_1) - \dot{h}_1(x_1, y_1)| = \mathcal{O}(Y'^{-3/2})$$

$$(2.10c) \quad |r_{\ell'}(x_1)| \leq 3A \cdot 4K_1^{-2} + \mathcal{O}(Y'^{-1/2}).$$

Therefore, by taking K_1 sufficiently large and assuming y_* large enough, we can ensure that equations (2.3) hold and that $\rho_{\ell'}$ is regular.

On the other hand if $(x_0, y_0) \in \tilde{\mathbf{C}}_1 \setminus \mathbf{C}_2^*$ we have, once more by Lemma 2.1 and part (a):

$$\begin{aligned} |h_{\ell'}(x_1) - h_1(x_1, y_1)| &\leq 2\bar{K}_2^{-1}Y'^{-1} \\ |\dot{h}_{\ell'}(x_1) - \dot{h}_1(x_1, y_1)| &= 3A \cdot 2\bar{K}_2^{-1} + \mathcal{O}(Y'^{-2}) \\ |r_{\ell'}(x_1)| &\leq 3A \cdot \mathcal{L}_{\ell'}^{-2}Y' + \mathcal{O}(1), \end{aligned}$$

from which we obtain that $\tilde{h}_{\ell'} \neq 0$, and $|\mathcal{L}_{\ell'}| > 1/2K_1Y'^{1/2}$ so that we can apply Lemma 2.1 to ℓ' and obtain:

(2.11a)

$$|h_{\ell''}(x_2) - h_1(x_2, y_2)| \leq 2K_1^{-1}Y'^{-3/2}$$

(2.11b)

$$|\dot{h}_{\ell''}(x_2) - \dot{h}_1(x_2, y_2)| = \mathcal{O}(Y'^{-3/2})$$

(2.11c)

$$|r_{\ell''}(x_2)| \leq 3AY'\mathcal{L}_{\ell'}^{-2}\mathcal{L}_{\ell'}^{-1} + 3A \cdot 2\bar{K}_2^{-1}Y'\mathcal{L}_{\ell'}^{-2} + \mathcal{O}(Y'^{-1/2}) \leq$$

(2.11d)

$$\leq 3A \cdot 4\bar{K}_2^{-1}(K_2^{-1} + 2K_1^{-2}) + \mathcal{O}(Y'^{-1/2})$$

which agree with equations (2.3) and prove that $\rho_{\ell''}$ is regular provided we take large enough K_1 , K_2 and y_* .

In order to conclude we need to carefully consider several possibilities: first assume that $\Gamma_{\ell} \cap \mathbf{C}_1 = \emptyset$ and cut the image of Γ_{ℓ} in as many \mathcal{J} -aligned curves as possible; in doing so we might be left with two boundary curves, that we denote by ℓ_-^* and ℓ_+^* . Consider for instance ℓ_-^* : there are two possibilities; if $|I_{\ell_-^*}| > \delta/4$ we can simply let $\ell_- = \ell_-^*$; otherwise we let ℓ_- be the union of ℓ_-^* with the adjacent pair; since the latter is \mathcal{J} -aligned, we obtain that ℓ_- is a standard pair since $|I_{\ell_-}| < 3/4\delta$; performing the same construction with ℓ_+^* we can conclude with:

$$F\ell = \bigcup_{\alpha \in \mathcal{A}} \bigcup_j \ell_{\alpha}^j \cup \ell_+ \cup \ell_-,$$

which concludes the proof of item (b) assuming that $\Gamma_{\ell} \cap \mathbf{C}_1 = \emptyset$.

Assume now that $\Gamma_{\ell} \cap \mathbf{C}_1 \neq \emptyset$; then by our choice of δ we know that $\Gamma_{\ell} \setminus \mathbf{C}_1$ has at most two connected components, that we denote by Γ_1 and Γ_2 ; in turn let $\Gamma_* = \Gamma_{\ell} \cap \mathbf{C}_1$. We will consider Γ_1 and Γ_2 separately; to fix ideas let us work with Γ_1 . Assume first that $|I_1| > 4\pi K_1^{-1}Y'^{-1/2}$; then, as before, we can cut the image of Γ_1 in as many \mathcal{J} -aligned curves as possible plus two boundary curves. One of them will not contain the image of $\partial\mathbf{C}_1$ whereas the other one will necessarily do. As before, we let the former to be ℓ_- , joining it with the adjacent one if it turns out to be too short; the preimage of the latter will be instead joined to Γ_* ; if, on the other hand $|I_1| \leq 4\pi K_1^{-1}Y'^{-1/2}$, then we join the whole Γ_1 to Γ_* . We do the same with the other connected component. Thus, as before we have

$$F\ell = \bigcup_{\alpha \in \mathcal{A}} \bigcup_j \ell_{\alpha}^j \cup \ell_+ \cup \ell_- \cup F\ell_*$$

and we are left with Γ_* such that $|I_*| > 3/2\delta K_1^{-1}Y'^{-1/2}$. By taking \hat{K}_1 sufficiently large we can ensure that $\ell_* \subset \hat{\mathbf{C}}_1$. Consider now $\Gamma_* \setminus \mathbf{C}_2^*$; by Lemma 2.9, this set has a uniformly bounded number of connected components; consider each connected component. If it is longer than $2K_2^{-1}Y'^{-1}$, we let its image be one of the $\tilde{\ell}_j$; by our previous arguments the image of $\tilde{\ell}_j$ can indeed be decomposed in standard pairs. We thus choose Δ so large that all short components will belong to $\hat{\mathbf{C}}_2$, which allows us to conclude. \square

We now introduce the notion of *critical time*; for a fixed standard partition \mathcal{J} , for any standard pair ℓ the critical time of a point $p \in \Gamma_\ell$ is the largest number \bar{n} such that, by iterating the decomposition in lemma 2.11, $F^n p$ belongs to a non-invalid curve for all $n \leq \bar{n}$.

Definition 2.12. Fix a standard partition \mathcal{J} and let ℓ be a standard pair. We define the critical time as a function $\tau_\ell : \Gamma_\ell \rightarrow \mathbb{N} \cup \{\infty\}$ obtained by means of the following recursive definition: let $p \in \Gamma_\ell$, then by item (b) of lemma 2.11 we have three possibilities:

- Fp belongs to a standard pair ℓ' : we then define $\tau_\ell(p) = \tau_{\ell'}(Fp) + 1$;
- Fp belongs to a stand-by pair, hence F^2p belongs to a standard pair ℓ'' : we define $\tau_\ell(p) = \tau_{\ell''}(F^2p) + 2$;
- otherwise we define $\tau_\ell(p) = 0$.

The following proposition is the crucial technical result of our work.

Proposition 2.13. If $\gamma > 2$, for any standard pair ℓ , we have $\mathbb{P}_\ell(\tau_\ell < \infty) = 1$.

The proof will be given in Section 4. We will now show how it implies our Main Theorem; the argument is a trivial adaptation of the analogous one found in [3]; we give it here for completeness.

Proof of the Main Theorem. First of all notice that Lebesgue measure can be disintegrated in reference pairs, i.e. for any E Borel measurable set:

$$\text{Leb}(E) = \int \mathbb{P}_{\ell_\alpha}(E) d\lambda_\alpha$$

where $d\lambda_\alpha$ is some factor measure on reference pairs. Furthermore, notice that by definition of τ_ℓ and by (2.8), Lemma 2.13 immediately implies that:

$$\mathbb{P}_\ell(\{(x_0, y_0) \in \Gamma_\ell \text{ s.t. } (x_n, y_n) \notin \hat{\mathbf{C}}_2 \ \forall n \in \mathbb{N}\}) = 0.$$

Hence we obtain:

$$(2.12) \quad \text{Leb}(\{(x_0, y_0) \text{ s.t. } (x_n, y_n) \notin \hat{\mathbf{C}}_2 \ \forall n \in \mathbb{N}\}) = 0.$$

Define now $\hat{F} : \hat{\mathbf{C}}_2 \rightarrow \hat{\mathbf{C}}_2$ as the first return map of F on $\hat{\mathbf{C}}_2$; \hat{F} is well defined almost everywhere by (2.12); moreover, lemma 2.9 implies that $\text{Leb}(\hat{\mathbf{C}}_2) < \infty$, consequently we can apply Poincaré recurrence theorem and conclude that almost every point in $\hat{\mathbf{C}}_2$ is recurrent, which shows that $\text{Leb}(\mathcal{E} \cap \hat{\mathbf{C}}_2) = 0$. This implies our Main Theorem since, using once more (2.12), we know that the orbit of almost every point in \mathbb{A}_* intersects $\hat{\mathbf{C}}_2$. \square

3. EQUIDISTRIBUTION

In this section we set up an induction scheme to prove equidistribution estimates on standard pairs for a specific class of observables. The observables we consider are sufficiently smooth function of the fast variable x which are constant on the y direction.

In the sequel, we will often need to approximate integrals of such observables with Riemann sums (or viceversa) over partitions which are highly non-uniform. Most element of the partition will have small size compared to a much smaller portion of them which have sizes that are order of magnitudes larger. The naïve bound on the Riemann sum, which is optimal for uniform partitions, gives estimates which are not sufficient for our purposes. The following lemma¹ will be systematically used to obtain crucial estimates.

Lemma 3.1. *Let (Ω, μ) be a finite measure space and $f : \Omega \rightarrow [0, 1]$ a measurable function. Assume there exist real numbers $0 < \lambda < 1$, $C > 0$ and $0 < \alpha \leq 1$ such that for any $1 \leq z \leq \lambda^{-1}$:*

$$\mu\{f > z\lambda\} \leq \mu(\Omega)Cz^{-\alpha}.$$

Then:

$$\mu(f) \leq \mu(\Omega)(C+1) \begin{cases} \frac{1}{1-\alpha}\lambda^\alpha & \text{if } \alpha < 1 \\ \lambda|\log \lambda| & \text{if } \alpha = 1. \end{cases}$$

Proof. Let $\hat{f} = \max(\lambda, f)$; then

$$\frac{\mu(f\lambda^{-1})}{\mu(\Omega)} \leq \frac{\mu(\hat{f}\lambda^{-1})}{\mu(\Omega)} \leq 1 + C \int_1^{\lambda^{-1}} z^{-\alpha} \leq 1 + C \begin{cases} \frac{\lambda^{\alpha-1}}{1-\alpha} & \text{if } \alpha < 1 \\ -\log \lambda & \text{if } \alpha = 1. \end{cases}$$

□

Given a standard pair ℓ , recall that we denote by \mathcal{L}_ℓ the expansion rate $\left| \frac{dx_1}{dx_0} \right|$ along Γ_ℓ and define:

$$\hat{\mathcal{L}}_\ell \doteq \inf_{\Gamma \setminus \mathbf{C}_1} \mathcal{L}_\ell;$$

moreover, define β so that $\gamma = 2\beta + 1$. We will often use the conventional notation $C_\#$ to indicate some positive real number which does not depend on y or other indices; the actual value of $C_\#$ can change from expression to expression.

Lemma 3.2 (Base equidistribution step). *There exists a constant C such that, given a standard pair ℓ , $\mathcal{A} \in \mathcal{C}(\mathbb{T}^1)$ and $B \in \mathcal{C}^1(\Gamma_\ell)$, we have:*

$$\begin{aligned} |\mathbb{E}_\ell(B \cdot \mathcal{A} \circ F) - \mathbb{E}_\ell(B) \langle \mathcal{A} \rangle| &\leq \|\mathcal{A}\| \left(\|B\| (\mathbb{P}_\ell(\mathbf{C}_1) + C\hat{\mathcal{L}}_\ell^{-1}) + \right. \\ (3.1) \quad &\left. + \|\dot{B}\| C\hat{y}_\ell^{-2\beta} \log \hat{y}_\ell \right) \end{aligned}$$

where $\mathcal{A} \circ F$ is a shorthand notation for $\mathcal{A}(\pi F(x, \psi_\ell(x)))$ and $\langle \mathcal{A} \rangle = \int_0^{2\pi} \mathcal{A}(\theta) d\theta$.

¹The original proof of this lemma was substantially more involved; I am, again, indebted to D. Dolgopyat for providing me with the much more elegant argument which is used here.

Notice that by linearity of expectation we can always assume that \mathcal{A} has zero average. The lemma ensures that, in one step, the dynamics acts on standard pairs by making them approach Lebesgue measure for observables which are independent of y . The lemma will be proved by means of the following slightly more general version;

Lemma 3.3. *Let I be a standard interval and ρ a regular probability density on I associated to the probability measure \mathbb{P} ; let $\varphi : I \rightarrow \mathbb{R}$ be a smooth function with at most one non-degenerate critical point and normalized so that $\|\dot{\varphi}\| \leq 1$; for $\mathcal{L} \gg 1$ sufficiently large, let $\Theta : I \rightarrow \mathbb{T}$ given by $\Theta = \mathcal{L}\varphi \bmod 2\pi$. Let $D = \{|\dot{\Theta}| < \mathcal{L}^{1/2}\}$ and $\hat{\mathcal{L}} = \inf_{I \setminus D} |\dot{\Theta}(x)| \leq \mathcal{L}$. Then there exist $C > 0$ which does not depend on \mathcal{L} , such that for any $B \in \mathcal{C}^1(I)$, $\mathcal{A} \in \mathcal{C}(\mathbb{T}^1)$ a zero average function:*

$$(3.2) \quad \left| \int_I B(x) \mathcal{A}(\Theta(x)) \rho(x) dx \right| \leq \|\mathcal{A}\| \left(\|B\| (\mathbb{P}(D) + C \hat{\mathcal{L}}^{-1}) + \|\dot{B}\|_{I \setminus D} C \mathcal{L}^{-1} \log \mathcal{L} \right).$$

Proof. Cut $I \setminus D$ at the points $\Theta = 0 \bmod 2\pi$ and denote by $\{J_k\}$ the set of intervals in $I \setminus D$ bounded by two consecutive cutting points. We obtain the following bound for the leftover pieces:

$$(3.3) \quad \mathbb{P}(D) \leq \mathbb{P}(I \setminus \bigcup_k J_k) \leq \mathbb{P}(D) + K \hat{\mathcal{L}}^{-1}.$$

The left inequality is obvious; for the right one notice that the number of leftover (connected) pieces is bounded by twice the number of connected components of $I \setminus D$, that can be at most 2 by definition of D . The measure of each of such pieces can in turn be bounded using Lemma 2.4 to obtain (3.3) with $K = 4\mu_2$.

Let

$$E_k = \int_{J_k} B(x) \mathcal{A}(\Theta(x)) \rho(x) dx;$$

for each k , let $\xi_k(\theta)$ be the inverse function of Θ on J_k ; moreover define the pushforward $\rho'_k(\theta) = \rho(\xi_k(\theta)) / |\dot{\Theta}(\xi_k(\theta))|$ and the auxiliary function given by $H_k(\theta) = B(x(\theta)) \rho'_k(\theta)$. Then:

$$E_k = \int_0^{2\pi} H_k(\theta) \mathcal{A}(\theta) d\theta.$$

Write $H_k(\theta) = \bar{H}_k + \tilde{H}_k(\theta)$, where \bar{H}_k is the average of H_k . Since \mathcal{A} has zero average we obtain:

$$|E_k| = \left| \int_0^{2\pi} \tilde{H}_k(\theta) \mathcal{A}(\theta) d\theta \right| \leq 2\pi \|\mathcal{A}\| \|\tilde{H}_k\|.$$

Fix k and let θ_0 be such that $\tilde{H}_k(\theta_0) = 0$, then:

$$|\tilde{H}_k(\theta)| \leq \int_{\theta_0}^{\theta} \left| \frac{d\tilde{H}_k}{ds} \right| ds \leq \int_{\theta_0}^{\theta} \rho'_k \left(\left| B \right| \left| \frac{d \log \rho'_k}{d\theta} \right| + \left| \frac{\dot{B}}{\dot{\Theta}} \right| \right) ds;$$

thus, if we let $c_k = \mathbb{P}(J_k)$ we obtain:

$$(3.4) \quad |E_k| \leq 2\pi \|\mathcal{A}\| c_k \left(\|B\|_{I \setminus D} \|r'\|_{J_k} + \|\dot{B}\|_{I \setminus D} \|\dot{\Theta}^{-1}\|_{J_k} \right).$$

where $r' = r\dot{\Theta}^{-1} + \ddot{\Theta}\dot{\Theta}^{-2}$; let $\hat{\mathcal{L}}_k = \inf_{x \in J_k} |\dot{\Theta}(x)| \geq \hat{\mathcal{L}}$, so that:

$$(3.5) \quad \|r'\|_{J_k} \leq \|r\|_{J_k} \hat{\mathcal{L}}_k^{-1} + \|\ddot{\Theta}\|_{J_k} \hat{\mathcal{L}}_k^{-2} \leq C_{\#}(\hat{\mathcal{L}}_k^{-1} + \mathcal{L} \hat{\mathcal{L}}_k^{-2}).$$

We thus need to obtain a bound for the two sums $\sum_k c_k \hat{\mathcal{L}}_k^{-1}$ and $\sum_k c_k \hat{\mathcal{L}}_k^{-2}$ and to this purpose we are going to use Lemma 3.1; on the one hand, for any c , if $\mathcal{L}_k > c\mathcal{L}$, then the two sums are bounded above by $C_{\#}\mathcal{L}^{-1}$ and $C_{\#}\mathcal{L}^{-2}$ respectively. On the other hand, since critical points of φ are non degenerate, there exist $c_{\varphi}, C_{\varphi} > 0$ such that, if $|\dot{\varphi}| < 2c_{\varphi}$, then $\dot{\varphi}$ is monotone and for any $z \geq 1$:

$$\mathbb{P}(|\dot{\varphi}| < c_{\varphi} z^{-1}) \leq C_{\varphi} z^{-1}$$

Since the diameter of each J_k is bounded above by $C_{\#}\hat{\mathcal{L}}^{-1}$, and using monotonicity of $\dot{\varphi}$ and boundedness of $\ddot{\varphi}$, the above estimate implies:

$$(3.6) \quad \mathbb{P}(\hat{\mathcal{L}}\hat{\mathcal{L}}_k^{-1} > \hat{\mathcal{L}}\mathcal{L}^{-1}c_{\varphi}^{-1}z) \leq C_{\varphi}z^{-1} + C_{\#}\hat{\mathcal{L}}^{-1}.$$

Let now Ω be the finite measure space whose elements are the intervals J_k with measure c_k ; let $f : J_k \mapsto \hat{\mathcal{L}}\hat{\mathcal{L}}_k^{-1}$; then we can apply Lemma 3.1 to f , since by construction $f(J_k) \in [0, 1]$ and (3.6) implies that

$$\mathbb{P}(f > \lambda z) \leq C_{\varphi}z^{-1} + C_{\#}\hat{\mathcal{L}}^{-1} \leq C_{\#}z^{-\alpha} \text{ with } \alpha = 1$$

with $\lambda = \hat{\mathcal{L}}\mathcal{L}^{-1}c_{\varphi}^{-1}$, where the second inequality holds because, by construction, $z^{-1} \geq \lambda = C_{\#}\mathcal{L}^{-1/2}$. We therefore obtain:

$$\sum_k c_k \hat{\mathcal{L}}_k^{-1} \leq C_{\#}\mathcal{L}^{-1} \log \mathcal{L}$$

Similarly, we apply Lemma 3.1 to f^2 , using $\lambda = \hat{\mathcal{L}}^2\mathcal{L}^{-2}c_{\varphi}^{-2}$ and $\alpha = 1/2$ obtaining:

$$\sum_k c_k \hat{\mathcal{L}}_k^{-2} \leq C_{\#}\mathcal{L}^{-1} \hat{\mathcal{L}}^{-1}.$$

Plugging the above estimates in (3.4) and using (3.5), we obtain (3.2) and finally conclude the proof. \square

Proof of Lemma 3.2. Let $\Theta_{\ell}(x) = x + Y_{\ell}(x)$; then we have by definition:

$$\mathbb{E}_{\ell}(B \cdot \mathcal{A} \circ F) = \int_{I_{\ell}} B(x, \psi_{\ell}(x)) \cdot \mathcal{A}(\Theta_{\ell}(x)) \rho_{\ell}(x) dx.$$

Let $\mathcal{L} = C_{\#}\hat{y}_{\ell}^{2\beta}$ and define $\varphi = \Theta_{\ell}/\mathcal{L}$; the reader will not find difficult to prove that φ satisfies the hypotheses of Lemma 3.3, which implies our Lemma. \square

Corollary 3.4. *There exists a constant C such that, for any ℓ , \mathcal{A} and B as in the statement of lemma 3.2 and $n \geq 0$ we have:*

$$(3.7) \quad \begin{aligned} & |\mathbb{E}_{\ell}(B \cdot \mathcal{A} \circ F^{n+1} 1_{\tau \geq n}) - \mathbb{E}_{\ell}(B \cdot 1_{\tau \geq n}) \langle \mathcal{A} \rangle| \leq \\ & \leq \|\mathcal{A}\| \left(\|B\| C \hat{y}_{\ell}^{-\beta} + \left\| \frac{dB}{dx_n} \right\|_* C \hat{y}_{\ell}^{-2\beta} \log \hat{y}_{\ell} \right). \end{aligned}$$

where the norm $\|\cdot\|_*$ is the sup restricted on those points which are mapped to a standard pair after n iterates.

Proof. If $n = 0$, the corollary trivially follows from lemma 3.2; we henceforth assume that $n > 0$ and, as before, that $\langle \mathcal{A} \rangle = 0$. Iterate lemma 2.11 for n times and obtain:

$$F^n \ell = \bigcup_j \ell'_j \cup \bigcup_k \tilde{\ell}_k \cup \{\tau < n\};$$

moreover we know that $F^{-1} \tilde{\ell}_k \subset \ell^* \cap \hat{\mathbf{C}}_1$ where ℓ^* is standard. Thus, by Lemma 2.9 we have:

$$\mathbb{P}_\ell(F^{-n} \bigcup_k \tilde{\ell}_k) \leq C_\# \cdot \hat{y}_\ell^{-\beta}.$$

We can hence obtain (3.7) by applying Lemma 3.2 to each standard pair ℓ'_j . \square

Remark 3.5. *If we choose B equal to the constant function 1 we obtain, by Lemma 3.2 and Corollary 3.4 that there exists a $C > 0$ such that:*

$$(3.8a) \quad |\mathbb{E}_\ell(\mathcal{A} \circ F) - \langle \mathcal{A} \rangle| \leq \|\mathcal{A}\| (\mathbb{P}_\ell(\mathbf{C}_1) + C \mathcal{L}_\ell^{-1} \log \hat{y}_\ell).$$

$$(3.8b) \quad |\mathbb{E}_\ell(\mathcal{A} \circ F^n 1_{\tau \geq n-1}) - \langle \mathcal{A} \rangle \mathbb{P}_\ell(\tau \geq n-1)| \leq \|\mathcal{A}\| C \hat{y}_\ell^{-\beta}.$$

We need to perform a substantially more accurate analysis in order to improve estimates (3.7). This is the principal technical difficulty of our work. Said analysis, which is the main result of this section, is summarized in the following

Lemma 3.6 (Equidistribution lemma). *For all $\beta > 1/2$ there exists $\nu(\beta) \in \mathbb{N}$ such that for any $n \geq \nu$, any sufficiently smooth function \mathcal{A} and any standard pair ℓ with \hat{y}_ℓ large enough, we have:*

$$(3.9) \quad |\mathbb{E}_\ell(\mathcal{A} \circ F^n 1_{\tau \geq n-1}) - \langle \mathcal{A} \rangle \mathbb{P}_\ell(\tau \geq n-1)| \leq \|\mathcal{A}\|' C_n o(\hat{y}_\ell^{-1}).$$

where $\|\mathcal{A}\|'$ is given by $\sum_l l^2 |\hat{\mathcal{A}}_l|$ with $\hat{\mathcal{A}}_l$ the l -th Fourier coefficient of \mathcal{A} and C_n is uniform in \hat{y}_ℓ .

Notice that if $\beta > 1$, then Lemma 3.6 immediately follows by Remark 3.5 by taking $\nu = 1$; this is essentially the work of [3]. In order to prove Lemma 3.6 for smaller values of β we will show the following result: if $\beta > 1/2$, then for $n \geq 2$, there exists a constant C_n such that for any sufficiently smooth function \mathcal{A} with $\langle \mathcal{A} \rangle = 0$ and standard pair ℓ as in the statement of Lemma 3.6 we have:

$$(3.10) \quad |\mathbb{E}_\ell(\mathcal{A} \circ F^n \cdot 1_{\tau \geq n-1})| \leq \|\mathcal{A}\|' C_n (\hat{y}_\ell^{-\beta - (n-1)(\beta - \frac{1}{2})} + o(\hat{y}_\ell^{-1})).$$

Lemma 3.6 then follows from (3.10) by choosing

$$\nu > \frac{1}{2} \left(\beta - \frac{1}{2} \right)^{-1}.$$

In the remaining part of this section we will always assume $1/2 < \beta \leq 1$; additionally, once β is fixed, we assume n to be fixed as well: in fact, our construction depends on n in that we are required to take ℓ with larger \hat{y}_ℓ as n grows and the constant C_n appearing in (3.10) tends to infinity as $n \rightarrow \infty$. This will not be an issue since we will invoke Lemma 3.6 with n bounded as a function of β .

Our proof of Lemma 3.6 is based on two main ingredients: the first one is an estimate of the contribution of the pieces of standard pairs which lie in \mathbf{C}_1 ; the second one is a cancellation estimate for higher iterates of F outside \mathbf{C}_1 . The former is in fact stated in lemma 3.14; the latter requires much finer estimates and will be described in the remaining part of this section.

We now introduce some convenient definitions: a basic pair ℓ is said to be a *clean pair* if $\Gamma_\ell \setminus \mathbf{C}_1$ is connected. Given ℓ and n we define a compact region of the phase space which contains $F^k \ell$ for $k \in \{0, \dots, n\}$. Let $\hat{y} = \hat{y}_\ell - 2A(n+1)$ and $\hat{y}^* = \hat{y}_\ell + 2A(n+2)$ and introduce the notation $\tilde{h}_S(x) = \tilde{h}_1(x, \hat{y})$; we will always assume \hat{y} to be large enough so that $\hat{y} < \hat{y}^* < 2\hat{y}$. Let $S = [\hat{y}, \hat{y}^*]$: we say that a standard pair ℓ is (S, k) -compatible if $F^l \Gamma_\ell \subset \mathbb{T} \times S$ for $0 \leq l \leq k$.

We now introduce a standard partition $\mathcal{J} = \{I_\alpha\}_{\alpha \in \mathcal{A}}$ which satisfies some useful properties:

- each I_α is such that $|\tilde{h}_S(x)|$ admits a unique minimum which we denote by \bar{x}_α ;
- if $x \in \text{int } I_\alpha$, then $\dot{\phi}(x) \neq 0$ and $\tilde{h}_S(x) \neq 0$;

In particular we have that any \mathcal{J} -adapted pair is a clean pair. Define the strips $S_\alpha = I_\alpha \times S$; on each strip we define adapted coordinates κ_α in such a way that $\kappa_\alpha(0, \eta) = (\bar{x}_\alpha, \eta)$. Let $\ell_\alpha^\eta = (\Gamma_\alpha^\eta, \bar{\rho}_\alpha = 1/|I_\alpha|)$ be the reference pair on the curve given by the image under κ_α of the horizontal line at height η , that is, the reference pair with base I_α passing through the point (\bar{x}_α, η) . For $\alpha \in \mathcal{A}$ let $\hat{\mathcal{L}}_\alpha = \inf_{\eta \in S} \hat{\mathcal{L}}_{\ell_\alpha^\eta}$ and for $1 \leq k \leq n$ define the following function:

$$(3.11) \quad \Psi_{\alpha,k}(\eta) = \mathbb{E}_{\ell_\alpha^\eta}(\mathcal{A} \circ F^k \cdot 1_{\tau \geq k-1}).$$

We now sketch the proof of (3.10): according to Lemma 2.11, a large portion of the image of a standard pair is given by a union of standard pairs; we need to prove that the weighted sum of the expectations of $\mathcal{A} \circ F^{n-1}$ over this union is of smaller order with respect to each term of the sum. In order to do so we need first to prove that we can approximate the expectation on a given standard pair with the expectation on an appropriate reference pair; hence we reduce to compute the weighted sum of the expectations on a number of reference pairs, that is, a weighted sum of a number of functions Ψ defined in (3.11). We will prove that Ψ are sufficiently regular and periodic in the variable $Y(\eta)$ up to a negligible error $\mathcal{O}(\hat{y}^{-1})$. This, and a fine control of the geometry of images of standard pairs allows us to prove an estimate for the cancellation at each step, which will finally lead to (3.10).

We now state a number of lemmata which will be used to prove (3.10); their proofs are quite technical and, as such, are postponed to the next subsection for easiness of exposition. We start with four lemmata related to the first iterate.

Lemma 3.7 (Comparison I). *There exists $C > 0$ such that for any standard pair ℓ and any reference pair $\bar{\ell}$ with $I_\ell = I_{\bar{\ell}}$ and $\Gamma_\ell \cap \Gamma_{\bar{\ell}} \neq \emptyset$ we have:*

$$|\mathbb{E}_\ell(\mathcal{A} \circ F) - \mathbb{E}_{\bar{\ell}}(\mathcal{A} \circ F)| \leq C \|\mathcal{A}\|_1 (\|r_\ell\| \hat{y}_\ell^{-\beta} + Y'(\hat{y}_\ell) \|\Delta h_\ell\|),$$

where $\|\cdot\|_1$ is the usual \mathcal{C}^1 -norm.

Lemma 3.8 (Periodicity). *Assume that $\eta_0, \eta_1 \in S$ with $Y(\eta_0) = Y(\eta_1) \pmod{1}$; then:*

$$(3.12) \quad |\Psi_{\alpha,1}(\eta_1) - \Psi_{\alpha,1}(\eta_0)| = \|\mathcal{A}\|_1 o(\hat{y}^{-1}).$$

Lemma 3.9 (Differentiability). *Assume that $\eta \in S$; then there exists $C > 0$ satisfying:*

$$(3.13) \quad \left| \frac{d\Psi_{\alpha,1}(\eta)}{d\eta} \right| \leq C \|\mathcal{A}\|_1 Y'(\eta) \hat{\mathcal{L}}_\alpha^{-1}.$$

Lemma 3.10 (Fourier components). *There exists $C > 0$ and a sequence $\{\hat{\Psi}_{\alpha,1}^{(l)}\}_{l \in \mathbb{Z}}$ such that, for all $\eta \in S$:*

$$(3.14) \quad \Psi_{\alpha,1}(\eta) = \sum_{l \in \mathbb{Z}} \hat{\Psi}_{\alpha,1}^{(k)} e^{-2\pi i l Y(\eta)} + \|\mathcal{A}'\|_1 o(\hat{y}^{-1}).$$

where $\hat{\Psi}_{\alpha,1}^{(0)} = 0$ and if $k \neq 0$:

$$(3.15) \quad |\hat{\Psi}_{\alpha,1}^{(k)}| \leq C |\mathcal{A}'_k| \hat{y}^{-\beta}$$

We proceed with three analogous lemmata related to higher iterates.

Lemma 3.11. *For all $2 \leq k \leq n$, there exists a constant C_k such that for all $\eta_1, \eta_2 \in S$ with $|Y(\eta_1) - Y(\eta_2)| < 1$ we have:*

$$(3.16) \quad |\Psi_{\alpha,k}(\eta_1) - \Psi_{\alpha,k}(\eta_2)| \leq \|\mathcal{A}\|_1 C_k o(\hat{y}^{-1}).$$

Lemma 3.12 (Comparison II). *For all $2 \leq k \leq n$ there exists C_k such that for any ℓ clean standard pair $(S, n-k)$ -compatible, there exists a reference pair $\bar{\ell}$ such that $I_\ell = I_{\bar{\ell}}$ satisfying*

$$(3.17) \quad \begin{aligned} & |\mathbb{E}_\ell(\mathcal{A} \circ F^k \cdot 1_{\tau \geq k-1}) - \mathbb{E}_{\bar{\ell}}(\mathcal{A} \circ F^k \cdot 1_{\tau \geq k-1})| \leq \\ & \leq \|r_\ell\| \mathbb{E}_{\bar{\ell}}(\mathcal{A} \circ F^k \cdot 1_{\tau \geq k-1}) + \|\mathcal{A}\| C_k o(\hat{y}^{-1}). \end{aligned}$$

Lemma 3.13 (Periodicity II). *There exist a subset $\mathcal{A}^* \subset \mathcal{A}$, constants ω_{α^*} where $\alpha^* \in \mathcal{A}^*$, constants C_k where $2 \leq k \leq n$ and sequences of coefficients $\hat{\Psi}_{\alpha,n}^{(\alpha^*, l)}$, where $l \in \mathbb{Z}$ and $\alpha^* \in \mathcal{A}^*$ such that for all $\eta \in S$ we have:*

$$(3.18) \quad \Psi_{\alpha,k}(\eta) = \sum_{l, \alpha^*} \hat{\Psi}_{\alpha,k}^{(\alpha^*, l)} e^{2\pi i l \omega_{\alpha^*} Y(\eta)} + \|\mathcal{A}'\|' o(\hat{y}^{-1}).$$

where there exists C'_k such that for all $l \in \mathbb{Z}$:

$$(3.19a) \quad |\hat{\Psi}_{\alpha,2}^{(\alpha^*, l)}| \leq C'_2 \left(\hat{y}^{\frac{1}{2}-\beta} + l \cdot \hat{y}^{1-2\beta} \log \hat{y} \right) \max_{\alpha' \in \mathcal{A}} |\hat{\Psi}_{\alpha',1}^{(l)}|;$$

$$(3.19b) \quad |\hat{\Psi}_{\alpha,k}^{(\alpha^*, l)}| \leq C'_k \left(\hat{y}^{\frac{1}{2}-\beta} + l \cdot \hat{y}^{-1} \right) \max_{\alpha' \in \mathcal{A}} |\hat{\Psi}_{\alpha',k-1}^{(\alpha^*, l)}| \quad \forall 2 < k \leq n;$$

this implies that:

$$(3.20) \quad |\hat{\Psi}_{\alpha,k}^{(\alpha^*, l)}| \leq C_k |\mathcal{A}'_l| l \cdot (\hat{y}^{-\beta-(k-1)(\beta-\frac{1}{2})} + o(\hat{y}^{-1})).$$

Moreover ω_{α^*} are of order \hat{y}^{-1} , that is there exist C such that

$$(3.21) \quad C^{-1} \hat{y}^{-1} < \omega_{\alpha^*} < C \hat{y}^{-1}.$$

We now show how, given the above lemma, we can obtain (3.10); in fact by Lemma 3.12 we have that:

$$|\mathbb{E}_\ell(\mathcal{A} \circ F^n \cdot 1_{\tau \geq n-1})| \leq 2|\mathbb{E}_{\tilde{\ell}}(\mathcal{A} \circ F^n \cdot 1_{\tau \geq n-1})| + \|\mathcal{A}\| C_n o(\hat{y}^{-1})$$

then using (3.18) and (3.20) we conclude that

$$|\mathbb{E}_{\tilde{\ell}}(\mathcal{A} \circ F^n \cdot 1_{\tau \geq n-1})| \leq \|\mathcal{A}\|' (C_n \hat{y}^{-\beta - (n-1)(\beta - \frac{1}{2})} + o(\hat{y}^{-1}))$$

from which (3.10) immediately follows.

3.1. Proofs of Lemmata 3.7-3.13. First of all notice that applying Lemma 2.11 to ℓ and the standard partition \mathcal{J} , we can decompose the image of ℓ as:

$$F\ell = \bigcup_{\alpha \in \mathcal{A}} \bigcup_{j \in \mathcal{J}_\alpha} \ell_\alpha^j \cup \ell_+ \cup \ell_- \cup \bigcup_j \tilde{\ell}_j \cup \{\tau = 0\}$$

We begin with the following proposition, which allows to control the contribution to $\mathbb{E}_\ell(\mathcal{A} \circ F^n \cdot 1_{\tau \geq n-1})$ given by non-aligned or non-standard pairs. As pointed out before, this proposition is crucial, as it deals with the dynamics inside the first order critical set $\hat{\mathbf{C}}_1$, and it is, loosely speaking, the counterpart of the base equidistribution step (Lemma 3.2) for curves intersecting $\hat{\mathbf{C}}_1$.

Proposition 3.14. *For any $2 \leq k \leq n$ and any $(S, n-k)$ -compatible standard pair ℓ , for any function $\mathcal{A} \in \mathcal{C}(\mathbb{T}^1)$ with zero average we have:*

$$(3.22) \quad \left| \mathbb{E}_\ell(\mathcal{A} \circ F^k \cdot 1_{\tau \geq k-1}) - \sum_{\alpha, j} c_\alpha^j \mathbb{E}_{\ell_\alpha^j}(\mathcal{A} \circ F^{k-1} \cdot 1_{\tau \geq k-2}) \right| \leq \|\mathcal{A}\| C_k o(\hat{y}^{-1}),$$

where $c_\alpha^j = \mathbb{P}_\ell(F^{-1} \ell_\alpha^j)$.

Proof. Since $F^{-1} \ell_\pm$ does not intersect \mathbf{C}_1 we have that $\mathbb{P}_\ell(F^{-1} \ell_\pm) = \mathcal{O}(\hat{y}^{-\beta})$, thus, by Corollary 3.4, the contribution of the two non-aligned standard curves is $\mathcal{O}(\hat{y}^{-2\beta}) = o(\hat{y}^{-1})$. Consequently, we only need to consider the contribution of stand-by pairs: if $k > 2$ we can conclude by a similar argument: decompose the image of stand-by pairs in standard pairs which we denote by $\{\ell_j''\}$; then:

$$\sum_j \mathbb{E}_{\tilde{\ell}_j}(\mathcal{A} \circ F^{k-1} \cdot 1_{\tau \geq k-2}) = \sum_j c_j'' \mathbb{E}_{\ell_j''}(\mathcal{A} \circ F^{k-2} \cdot 1_{\tau \geq k-3}),$$

where $c_j'' = \mathbb{P}_\ell(F^{-2} \Gamma_j'')$. Now we can apply Corollary 3.4 to the right hand side and conclude, since $\sum_j c_j'' \leq \mathbb{P}_\ell(\hat{\mathbf{C}}_1) = \mathcal{O}(\hat{y}^{-\beta})$ by Lemma 2.9.

It remains to prove the case $k = 2$: we apply the scheme of the proof of Lemma 3.3; let us denote by \tilde{I} the base of the preimage of a connected component $\tilde{\ell}$ of the stand-by portion; Lemma 2.9 implies that we have a uniformly bounded number of connected components it suffices to prove that the contribution of each component is $o(\hat{y}^{-1})$. First of all notice that we necessarily have $\tilde{I} \subset \{\tau \geq 1\}$; then, for $x_0 \in I$, define $\Theta(x_0) = x_0 + Y(y_0) + Y(y_1)$; cut \tilde{I} at the points $\Theta = 0 \pmod{2\pi}$ and let $\{J_j\}$ denote the set of intervals in \tilde{I} bounded by two consecutive cutting points. Then applying (2.6c) we immediately obtain:

$$\mathbb{P}_\ell(\tilde{I} \setminus \bigcup_k J_k) = \mathcal{O}(\hat{y}^{-2\beta}).$$

Define

$$E_k = \int_{J_k} \mathcal{A}(\Theta(x)) \rho(x) dx;$$

then we can write:

$$\left| \int_{\bar{I}} \mathcal{A}(\Theta(x)) \rho(x) dx - \sum_k E_k \right| \leq \|\mathcal{A}\| o(\hat{y}^{-1}).$$

On each $F^2\Gamma_k$ we can define an inverse function for Θ and we can push forward the density ρ as $\rho_k''(\theta) = \rho(x(\theta))/|\dot{\Theta}(x(\theta))|$. Separating from the constant part we have:

$$|E_k| \leq 2\pi \|\mathcal{A}\| \|\tilde{\rho}_k''\| \leq 2\pi \|\mathcal{A}\| c_k \left\| \frac{d \log \rho_k''}{d\theta} \right\|.$$

The norm can be computed using (2.11c) which gives:

$$\|r_k''\| \leq \underbrace{\frac{6A}{K_2} \|\mathcal{L}_\ell^{-1}\|_{J_k}}_{X_k} + \underbrace{\frac{6A}{K_2} \|Y' \mathcal{L}_{\ell'}^{-2}\|_{J_k}}_{Y_k} + \mathcal{O}(\hat{y}^{-\beta}).$$

Again, let us consider the discrete measure space Ω whose elements are the intervals J_k , each of measure c_k ; note that $\mu(\Omega) = \mathcal{O}(\hat{y}^{-\beta})$. We then define $f : J_k \mapsto C_\# X_k$ so that we can apply Lemma 3.1 with $\lambda = C_\# \hat{y}^{-\beta}$ and $\alpha = 1$; the fact that f satisfies the hypotheses of Lemma 3.1 follows from the analysis we performed in the proof of Lemma 3.3. We thus obtain:

$$\sum_k c_k X_k = \mathcal{O}(\hat{y}^{-2\beta} \log \hat{y}) = o(\hat{y}^{-1}).$$

Similarly, let $g : J_k \mapsto C_\# Y_k$; we claim that the hypotheses of Lemma 3.1 hold with $\lambda = C_\# \hat{y}^{-2\beta}$ and $\alpha = 1$: this follows since if J_k is such that $Y' \mathcal{L}_{\ell'}^{-2} > C_\# \hat{y}^{-2\beta} z$, for some $z \geq 1$, then $\tilde{h}_{\ell'} < C_\# z^{-\frac{1}{2}}$, but since $\tilde{h}_\ell \tilde{h}_{\ell'} > K_2 Y'^{-1}$ we immediately obtain that $\mathcal{L}_\ell = Y' \tilde{h}_\ell > z^{\frac{1}{2}}$. We can thus bound the measure of such J_k by $\mathcal{O}(z^{-1})$ i.e. we can choose $\alpha_g = 1$. We can then conclude that

$$\sum_k c_k Y_k = o(\hat{y}^{-1}),$$

which implies (3.22) and concludes the proof. \square

Next, given a clean standard curve Γ we want to find a reference curve $\bar{\Gamma}$ such that the image of $F\bar{\Gamma}$ shadows $F\Gamma$ very closely; this will be obtained by means of the following

Lemma 3.15 (Shadowing by reference curves). *Let ℓ be a clean standard pair and let $\Gamma^* = \Gamma_\ell \setminus \mathbf{C}_1$. Then there exist a reference pair $\bar{\ell}$ such that $I_{\bar{\ell}} = I_\ell$ and a subset $\Gamma' \subset \Gamma^*$ such that $\mathbb{P}_\ell(\Gamma^* \setminus \Gamma') = \mathcal{O}(\hat{y}^{-3\beta})$ and:*

$$\forall (x_1, y_1) \in F\Gamma' \exists \bar{y}_1 \text{ s.t. } (x_1, \bar{y}_1) \in F\bar{\Gamma} \text{ and } |\bar{y}_1 - y_1| = \mathcal{O}(\hat{y}^{-5\beta}).$$

Proof. Define the slope field $h_{-1}(x_0, y_0) = -Y'(y_0)^{-1}$; then for any $(x_1, y_1) \in F\Gamma^*$ consider the vertical line $\{x = x_1\}$ passing through (x_1, y_1) ;

it is easy from the definitions to check that $F^{-1}\{x = x_1\}$ is an integral curve of h_{-1} ; moreover, again from the definition it is easy to obtain the relation

$$(3.23) \quad \left. \frac{dy_1}{dx_0} \right|_{h_{-1}} = Y'(y_0)^{-1}.$$

Let $I^* = \pi_x \Gamma^*$ and let $\bar{x} \in I^*$ such that $|\ddot{\phi}(\bar{x})| = \min_{x \in I^*} |\ddot{\phi}(x)|$; let $\bar{\Gamma} = (x, \bar{\psi}(x))$ be the reference curve over I passing through $(\bar{x}, \psi_\ell(\bar{x}))$ and $\bar{\rho}$ be the uniform density on $\bar{\Gamma}$. By Lemma 2.6 and the definition of standard curve, we have that the vertical distance between Γ and $\bar{\Gamma}$ is bounded by:

$$(3.24) \quad |\psi(x) - \bar{\psi}(x)| \leq C_\# |x - \bar{x}| \hat{y}^{-3\beta}.$$

Define

$\Gamma' = \{p \in \Gamma^* \text{ s.t. the integral curve of } h_{-1} \text{ passing through } p \text{ intersects } \bar{\Gamma}\}$;

then for each $p \in \Gamma'$ we define Π_p as the piece of integral curve of h_{-1} connecting Γ' to $\bar{\Gamma}$. The proof is then complete provided that we prove that $|\pi_x \Pi_p|$ is uniformly bounded in Γ' by $C_\# \cdot \hat{y}^{-3\beta}$ and then using (3.23). First obtain a rough upper bound:

$$(3.25) \quad |\pi_x \Pi_p| < \frac{\max_{x \in I^*} |\psi(x) - \psi(\bar{x})|}{\min_{x \in I^*} |h_\ell(x) + Y'^{-1}(y_\Pi(x))|} < C_\# \hat{y}^{-2\beta}.$$

Since $|\tilde{h}_\ell(x)| > 1/2 |\tilde{h}_\ell(\bar{x})| + C_\# |x - \bar{x}|$, estimates (3.25) and (3.24) allows us to obtain the better estimate:

$$|\pi_x \Pi_{(x_0, y_0)}| < 2 \frac{\max_{|x-x_0| < C_\# \hat{y}^{-2\beta}} |\psi(x) - \psi(\bar{x})|}{\min_{|x-x_0| < C_\# \hat{y}^{-2\beta}} |\tilde{h}_\ell(x)|} < C_\# \hat{y}^{-3\beta},$$

which concludes the proof. \square

We now proceed to give the proofs of Lemmata 3.7-3.10

Proof of Lemma 3.7. Fix $\bar{x} \in I$, define $\bar{\Gamma} = (x, \bar{\psi}(x))$ to be the reference curve over I passing through the point $(\bar{x}, \psi_\ell(\bar{x}))$, let $\bar{\rho}$ be the uniform density on I and define $\ell^* = (\Gamma_\ell, \bar{\rho})$ and $\bar{\ell} = (\bar{\Gamma}, \bar{\rho})$. Then we can write:

$$(3.26a) \quad |\mathbb{E}_\ell(\mathcal{A} \circ F) - \mathbb{E}_{\bar{\ell}}(\mathcal{A} \circ F)| \leq |\mathbb{E}_\ell(\mathcal{A} \circ F) - \mathbb{E}_{\ell^*}(\mathcal{A} \circ F)| +$$

$$(3.26b) \quad + |\mathbb{E}_{\ell^*}(\mathcal{A} \circ F) - \mathbb{E}_{\bar{\ell}}(\mathcal{A} \circ F)|.$$

We bound (3.26a) by applying lemma 3.2 to ℓ^* with $B = (\rho_\ell - \bar{\rho})/\bar{\rho}$. In fact it is easy to check that $\|B\| \leq \delta \|r_\ell\|$ and $\|\dot{B}\| \leq 2\|r_\ell\|$; hence we obtain

$$|\mathbb{E}_\ell(\mathcal{A} \circ F) - \mathbb{E}_{\ell^*}(\mathcal{A} \circ F)| \leq 2\|\mathcal{A}\| \|r_\ell\| C_\# \hat{y}^{-\beta}.$$

Introduce the functions $\Theta(x) = x + Y_\ell(x)$ and $\bar{\Theta}(x) = x + Y_{\bar{\ell}}(x)$; Lemma 2.6 implies that $|\psi_\ell(x) - \psi_{\bar{\ell}}(x)| \leq C_\# \|\Delta h_\ell\| |I|$, which yields:

$$\|\Theta - \bar{\Theta}\| \leq C_\# Y'(\hat{y}_\ell) \|\Delta h_\ell\|;$$

we can thus rewrite (3.26b) as:

$$\begin{aligned} |\mathbb{E}_{\ell^*}(\mathcal{A} \circ F) - \mathbb{E}_{\bar{\ell}}(\mathcal{A} \circ F)| &= \bar{\rho} \left| \int_I \mathcal{A}(\Theta(x)) - \mathcal{A}(\bar{\Theta}(x)) dx \right| \leq \\ &\leq \bar{\rho} \|\mathcal{A}'\| \|\Theta - \bar{\Theta}\| \leq C_\# \|\mathcal{A}\|_1 Y'(\hat{y}_\ell) \|\Delta h_\ell\|, \end{aligned}$$

which concludes the proof. \square

Proof of Lemma 3.8. To fix ideas, we consider $\eta_1 > \eta_0$ and let $I_\alpha = [a, b]$ assuming without loss of generality that $a = \bar{x}_\alpha$; moreover introduce the following shorthand notations: $\Gamma_i = \Gamma_{\ell_i}^{\eta_i}$, $\psi_i = \psi_{\ell_i}^{\eta_i}$, $Y_i = Y_{\ell_i}^{\eta_i}$ and similarly for Y' and Y'' . Define

$$\Theta_0(x) = x + Y_0(x) \quad \Theta_1(x) = x + Y_1(x) + Y(\eta_0) - Y(\eta_1)$$

so that $\Theta_0(a) = \Theta_1(a)$; let $\delta\Theta = \Theta_1 - \Theta_0$ and for $\lambda \in [0, 1]$ let $\Theta_\lambda = (1 - \lambda)\Theta_0 + \lambda\Theta_1$ so that $\partial_\lambda \Theta_\lambda = \delta\Theta$.

We claim that, for any $\lambda \in [0, 1]$ the following estimates hold:

$$(3.27a) \quad |\delta\Theta(x)| \leq \left| \frac{\dot{\Theta}_\lambda(x)^2}{Y'(\hat{y})} \right| |\eta_1 - \eta_0| \mathcal{O}(\hat{y}^{-1})$$

$$(3.27b) \quad |\delta\dot{\Theta}(x)| \leq |\dot{\Theta}_\lambda(x)| |\eta_1 - \eta_0| \mathcal{O}(\hat{y}^{-1})$$

In fact (3.27b) follows by direct computations, using the definition of Θ ; in order to prove (3.27a) write:

$$\delta\Theta(x) = \int_a^x \delta\dot{\Theta}(\xi) d\xi.$$

Notice that if $|\dot{\Theta}_\lambda(x)| > cY'(\hat{y})$, then (3.27a) immediately follows from (3.27b); otherwise, we know that $|\ddot{\Theta}_\lambda(x)| > c'Y'(\hat{y})$ and that for each $a \leq \xi \leq x$ we have $|\dot{\Theta}_\lambda(\xi)| \leq |\dot{\Theta}_\lambda(x)|$; therefore:

$$|\delta\Theta(x)| \leq |\delta\dot{\Theta}(x)| \left| \frac{\dot{\Theta}(x)}{c'Y'(\hat{y})} \right|$$

from which we conclude, again using (3.27b).

Define the function

$$(3.28) \quad \tilde{\Psi}_{\alpha,1}(\lambda) = \int_a^b \mathcal{A}(\Theta_\lambda(x)) \bar{\rho}_\alpha dx;$$

clearly $\tilde{\Psi}_{\alpha,1}(0) = \Psi_{\alpha,1}(\eta_0)$ and since $Y(\eta_1) - Y(\eta_0) = 0 \pmod{1}$ we obtain that $\tilde{\Psi}_{\alpha,1}(1) = \Psi_{\alpha,1}(\eta_1)$. We now claim that:

$$(3.29) \quad \frac{d\tilde{\Psi}_{\alpha,1}}{d\lambda} = \mathcal{A}(\Theta_\lambda(b)) \frac{\delta\Theta(b)}{\dot{\Theta}_\lambda(b)} + \|\mathcal{A}\|_1 o(\hat{y}^{-1})$$

from which we conclude; in fact:

$$\Psi_{\alpha,1}(\eta_1) - \Psi_{\alpha,1}(\eta_0) = \int_0^1 \frac{d\tilde{\Psi}_{\alpha,1}}{d\lambda} d\lambda = \int_0^1 \mathcal{A}(\Theta_\lambda(b)) \frac{\delta\Theta(b)}{\dot{\Theta}_\lambda(b)} \bar{\rho}_\alpha d\lambda + \|\mathcal{A}\|_1 o(\hat{y}^{-1});$$

notice that by definition $\dot{\Theta}_\lambda(b)^{-1}$ is a decreasing function of λ ; since \mathcal{A} has zero average we obtain the following bound:

$$\left| \int_0^1 \mathcal{A}(\Theta_\lambda(b)) \frac{\delta\Theta(b)}{\dot{\Theta}_\lambda(b)} \bar{\rho}_\alpha d\lambda \right| < \|\mathcal{A}\| \left| \frac{\delta\Theta(b)}{\dot{\Theta}_0(b)} \right| \bar{\lambda},$$

where $\bar{\lambda}$ is so that $\Theta_{\bar{\lambda}}(b) = \Theta_0(b) + 1$ or $\bar{\lambda} = 1$ if the previous equation has no solutions; since $\partial_\lambda \Theta(b) = \delta\Theta(b)$ we have $\bar{\lambda} = \min(1, \mathcal{O}(\delta\Theta(b)^{-1}))$, which implies

$$\left| \frac{\delta\Theta(b)}{\dot{\Theta}_0(b)} \right| \bar{\lambda} \leq C_\# |\dot{\Theta}_0(b)^{-1}| = o(\hat{y}^{-1})$$

and concludes the proof. We then need to prove (3.29): differentiate (3.28) with respect to λ and obtain

$$\frac{d\tilde{\Psi}_{\alpha,1}}{d\lambda} = \int_{I_\alpha} \mathcal{A}'(\Theta_\lambda(x)) \partial_\lambda \Theta_\lambda(x) \bar{\rho}_\alpha dx.$$

Let $J = \{x \in I_\alpha \text{ s.t. } |\dot{\Theta}_\lambda| \geq 1\}$; by construction of the partition, one of the boundary points of J , which we denote by a' is either equal to a or $\mathcal{O}(\hat{y}^{-2\beta})$ -close to a and the other one is necessarily b , hence $J = [a', b]$. We have by (3.27a) that $\delta\Theta(a') = \mathcal{O}(\hat{y}^{-1-2\beta})$ and we thus conclude that:

$$\int_a^b \mathcal{A}'(\Theta_\lambda(x)) \partial_\lambda \Theta_\lambda(x) dx = \int_{a'}^b \mathcal{A}'(\Theta_\lambda(x)) \partial_\lambda \Theta_\lambda(x) dx + o(\hat{y}^{-1})$$

We then integrate by parts and obtain:

$$\begin{aligned} \int_{a'}^b \mathcal{A}'(\Theta_\lambda(x)) \partial_\lambda \Theta_\lambda(x) dx &= \int_{a'}^b \mathcal{A}'(\Theta_\lambda(x)) \dot{\Theta}_\lambda(x) \frac{\delta\Theta(x)}{\dot{\Theta}_\lambda(x)} dx = \\ &= \mathcal{A}(\Theta_\lambda(x)) \frac{\delta\Theta(x)}{\dot{\Theta}_\lambda(x)} \Big|_{a'}^b - \int_{a'}^b \mathcal{A}(\Theta_\lambda(x)) \left(\frac{\delta\dot{\Theta}(x)}{\dot{\Theta}_\lambda(x)} - \frac{\delta\Theta(x)\ddot{\Theta}_\lambda(x)}{\dot{\Theta}_\lambda^2(x)} \right) dx. \end{aligned}$$

We first deal with the boundary terms; on the one hand the contribution of the term corresponding to a' is $\|\mathcal{A}'\| \mathcal{O}(\hat{y}^{-1-2\beta})$ which is negligible; on the other hand the term corresponding to b gives the main term in the right hand side of (3.29). We are thus left to show that the integral term is $o(\hat{y}^{-1})$; let:

$$B = \frac{\delta\dot{\Theta}}{\dot{\Theta}_\lambda} - \frac{\delta\Theta\ddot{\Theta}_\lambda}{\dot{\Theta}_\lambda^2}.$$

Then we conclude using Lemma 3.3 since by (3.27a) and (3.27b) we immediately have that $\|B\| = \mathcal{O}(\hat{y}^{-1})$ and $\|\dot{B}\|_{I/D} = \mathcal{O}(\hat{y}^{-1+\beta})$. \square

Proof of Lemma 3.9. Define $\Theta_\alpha(x, \eta) = x + Y(\psi_{\ell_\alpha^\eta}(x))$; by definition of $\psi_{\ell_\alpha^\eta}(x)$ we can write:

$$(3.30) \quad \partial_\eta \Theta_\alpha(x, \eta) = Y'(\eta) \frac{Y'(\eta - 2\dot{\phi}(\bar{x}_\alpha))}{Y'(\eta)} \frac{Y'(\psi_{\ell_\alpha^\eta}(x))}{Y'(\psi_{\ell_\alpha^\eta}(x) - 2\dot{\phi}(x))}$$

then by definition:

$$\Psi_{\alpha,1}(\eta) = \int_{I_\alpha} \mathcal{A}(\Theta_\alpha(x, \eta)) \bar{\rho}_\alpha dx$$

thus:

$$\frac{d\Psi_{\alpha,1}(\eta)}{d\eta} = \int_{I_\alpha} \mathcal{A}'(\Theta_\alpha(x, \eta)) \partial_\eta \Theta_\alpha(x, \eta) \bar{\rho}_\alpha dx$$

We conclude by applying Lemma 3.3 to the previous integral using $B = \partial_\eta \Theta_\alpha(x, \eta)$; in fact (3.30) implies that $\|B\| = Y'(\eta)(1 + \mathcal{O}(\hat{y}^{-1}))$ and $\|\dot{B}\| = Y'(\eta)\mathcal{O}(\hat{y}^{-1})$. \square

Proof of Lemma 3.10. As in the proof of Lemma 3.9 define $\Theta_\alpha(x, \eta) = x + Y(\psi_{\ell_\alpha^\eta}(x))$; fix $\bar{Y} = Y(\hat{y})$ for $\hat{y} \in S$ such that $\bar{Y} = 0 \pmod{1}$ and notice that

$$\Theta_\alpha(x, \eta(\bar{Y} + \theta)) = \Theta_\alpha(x, \eta(\bar{Y})) + \theta + \mu(\theta, x),$$

with $\|\mu\|_1 = \mathcal{O}(\hat{y}^{-1})$ by (3.30). For $k \in \mathbb{Z}$ define the following sequence:

$$\hat{\Psi}_{\alpha,1}^{(k)} = \bar{\rho}_\alpha \int_{I_\alpha} \hat{\mathcal{A}}_k e^{2\pi i k \Theta_\alpha(x, \eta(\bar{Y}))} dx$$

Notice that (3.15) follows by applying lemma 3.2 to the functions $\theta \mapsto \hat{\mathcal{A}}_k e^{2\pi i k \theta}$. Then we need to prove (3.14). Notice that lemma 3.8 implies that $\Psi_{\alpha,1}(\eta(Y))$ is periodic in Y up to $o(\hat{y}^{-1})$; therefore it suffices to show that (3.14) holds for $\eta \in [\eta(\bar{Y}), \eta(\bar{Y} + 1)]$. Notice that by definition:

$$\sum_{k \in \mathbb{Z}} \hat{\Psi}_{\alpha,1} e^{2\pi i k \theta} = \bar{\rho}_\alpha \int_{I_\alpha} \mathcal{A}(\Theta_\alpha(x, \eta(\bar{Y})) + \theta) dx,$$

and since \mathcal{A} is smooth:

$$\begin{aligned} \mathcal{A}(\Theta_\alpha(x, \eta(\bar{Y})) + \theta) &= \mathcal{A}(\Theta_\alpha(x, \eta(\bar{Y} + \theta))) + \\ &+ \mathcal{A}'(\Theta_\alpha(x, \eta(\bar{Y} + \theta)))\mu(\theta, x) + \|\mathcal{A}''\|\mathcal{O}(\mu^2). \end{aligned}$$

Consequently:

$$\begin{aligned} \Psi_{\alpha,1}(\eta(\bar{Y} + \theta)) - \sum_{k \in \mathbb{Z}} \hat{\Psi}_{\alpha,1} e^{2\pi i k \theta} &= - \int_{I_\alpha} \bar{\rho}_\alpha \mathcal{A}'(\Theta_\alpha(x, \eta(\bar{Y} + \theta)))\mu(\theta, x) dx + \\ &+ \|\mathcal{A}''\|\mathcal{O}(\mu^2) \end{aligned}$$

We claim that the right hand side is $o(\hat{y}^{-1})$, which proves (3.14): in fact by applying Lemma 3.3 to the first term we obtain a bound $\|\mathcal{A}'\|\mathcal{O}(\hat{y}^{-1-\beta} \log \hat{y})$; the second term can in turn be easily bounded since $\mu^2 = \mathcal{O}(\hat{y}^{-2})$. \square

We will prove Lemmata 3.11-3.13 by means of the following induction scheme: using Lemmata 3.7-3.10 we will prove Lemma 3.16 for $k = 2$, from which will follow Lemmata 3.11-3.13 for $k = 2$; then assuming we proved Lemmata 3.11-3.13 for k , we prove 3.16 for $k+1$ and thus Lemmata 3.11-3.13 for $k+1$.

The following Lemma is the base induction step which will be used in all the remaining proofs.

Proposition 3.16 (Base induction step). *For all $2 \leq k \leq n$ we have:*

(a) *let ℓ be any clean standard pair that is $(S, n-k)$ -compatible; then by proposition 3.14 we know that:*

$$\mathbb{E}_\ell(\mathcal{A} \circ F^k \cdot 1_{\tau \geq k-1}) = \sum_{\alpha, j} c_\alpha^j \mathbb{E}_{\ell_\alpha^j}(\mathcal{A} \circ F^{k-1} \cdot 1_{\tau \geq k-2}) + \|\mathcal{A}\|o(\hat{y}^{-1}).$$

For each α there exists an index set \mathbf{J}_α , which excludes at most a uniformly bounded number of indices, and $\{\eta_\alpha^j\}$ such that

$$(3.31) \quad \mathbb{E}_\ell(\mathcal{A} \circ F^k \cdot 1_{\tau \geq k-1}) = \sum_{\alpha} \sum_{j \in \mathbf{J}_\alpha} c_\alpha^j \Psi_{\alpha, k-1}(\eta_\alpha^j) + \|\mathcal{A}\|o(\hat{y}^{-1}),$$

where η_α^j satisfies the following estimate:

$$(3.32) \quad Y(\eta_\alpha^j) = Y(\eta_\alpha^0) + j - \theta_\alpha^j + \theta_\alpha^0 + \mu_\alpha(j - \theta_\alpha^j + \theta_\alpha^0)$$

where $\theta_\alpha^j \in I$ belongs to a $\mathcal{O}(\hat{y}^{-3\beta})$ -neighborhood of the point $\bar{\theta}_\alpha^j$, where $\bar{\theta}_\alpha^j$ is such that

$$\pi_x F(\bar{\theta}_\alpha^j, \psi(\bar{\theta}_\alpha^j)) = \bar{x}_\alpha$$

and μ_α is a function such that:

(3.33)

$$\mu_\alpha(0) = 0 \quad \mu'_\alpha = \left(\frac{Y'(\hat{y} + 2\dot{\phi}(\bar{x}_\alpha))}{Y'(\hat{y})} - 1 \right) + \mathcal{O}(\hat{y}^{-2}) \quad \mu''_\alpha = \mathcal{O}(\hat{y}^{-2-2\beta}).$$

(b) for any two clean standard pairs ℓ_1 and ℓ_2 , both $(S, n-k)$ -compatible and such that $\|Y \circ \psi_1 - Y \circ \psi_2\| < 1$, then for each α there exists a common index set \mathbf{J}_α satisfying:

$$\begin{aligned} \mathbb{E}_{\ell_1}(\mathcal{A} \circ F^k \cdot 1_{\tau \geq k-1}) &= \sum_{\alpha} \sum_{j \in \mathbf{J}_\alpha} c_{\alpha,1}^j \Psi_{\alpha,k-1}(\eta_{\alpha,1}^j) + \|\mathcal{A}'\| o(\hat{y}^{-1}) \\ \mathbb{E}_{\ell_2}(\mathcal{A} \circ F^k \cdot 1_{\tau \geq k-1}) &= \sum_{\alpha} \sum_{j \in \mathbf{J}_\alpha} c_{\alpha,2}^j \Psi_{\alpha,k-1}(\eta_{\alpha,2}^j) + \|\mathcal{A}'\| o(\hat{y}^{-1}) \end{aligned}$$

and the following estimate holds true:

$$(3.34) \quad |\eta_{\alpha,1}^j - \eta_{\alpha,2}^j| \leq 2 \frac{\|\psi_1 - \psi_2\|}{\tilde{h}(\theta_{\alpha,1}^j) Y'}.$$

Proof. By proposition 3.14 we know that

$$\mathbb{E}_\ell(\mathcal{A} \circ F^k \cdot 1_{\tau \geq k-1}) = \sum_{\alpha,j} c_{\alpha}^j \mathbb{E}_{\ell_\alpha^j}(\mathcal{A} \circ F^{k-1} \cdot 1_{\tau \geq k-2}) + \|\mathcal{A}\| o(\hat{y}^{-1}).$$

In order to obtain (3.31), consider the reference curve $\bar{\Gamma}$ with $\bar{\Gamma} = (x, \bar{\psi}(x))$ given by Lemma 3.15. We know that its image is $\mathcal{O}(\hat{y}^{-5\beta})$ -close to $F\Gamma^*$ along the vertical direction outside a small set of measure $\mathcal{O}(\hat{y}^{-3\beta})$ which we neglect. Hence we can find η_α^k s such that

$$(\bar{x}_\alpha, \eta_\alpha^k) \in F\bar{\Gamma} \text{ is close to } \Gamma_\alpha^k.$$

First we prove equation (3.32): by definition, a point (x, y) is in the preimage of $\{x = \bar{x}_\alpha\}$ if it satisfies the following equation:

$$(3.35) \quad x + Y(y) = \bar{x}_\alpha \pmod{1}$$

Therefore, by imposing $(x, y) \in \bar{\Gamma}$ we obtain an equation for the points

$$\theta_\alpha^k = \pi_x F^{-1}(\bar{x}_\alpha, \eta_\alpha^k);$$

since ℓ is a clean standard pair, we can write:

$$Y(\bar{\psi}(\theta_\alpha^k)) = \bar{x}_\alpha + N_\alpha + k - \theta_\alpha^k,$$

where N_α is such that $\theta_\alpha^{k=0}$ is the point satisfying (3.35) closest to \bar{x}_α and; we define \mathbf{J}_α as the set of k 's which satisfy the above equation. Notice that since ℓ is a clean standard pair we have either $\mathbf{J}_\alpha \subset \{k \leq 0\}$ or $\mathbf{J}_\alpha \subset \{k \geq 0\}$. Since we have

$$\eta_\alpha^k = \bar{\psi}(\theta_\alpha^k) + 2\dot{\phi}(\bar{x}_\alpha),$$

we define μ_α such that the following equation holds true:

$$\begin{aligned} Y(\eta_\alpha^k) &= Y(Y^{-1}(\bar{x}_\alpha + N_\alpha + k - \theta_\alpha^k) + 2\dot{\phi}(\bar{x}_\alpha)) = \\ &= Y(\eta_\alpha^0) + k - \theta_\alpha^k + \theta_\alpha^0 + \mu_\alpha(k - \theta_\alpha^k + \theta_\alpha^0). \end{aligned}$$

Clearly $\mu_\alpha(0) = 0$; by simple calculations we obtain:

$$\begin{aligned}\mu'_\alpha(t) &= \frac{Y'(Y^{-1}(\bar{x}_\alpha + N_\alpha + t - \theta_\alpha^0) + 2\dot{\phi}(\bar{x}_\alpha))}{Y'(Y^{-1}(\bar{x}_\alpha + N_\alpha + t - \theta_\alpha^0))} \\ \mu''_\alpha(t) &= \mathcal{O}(\hat{y}^{-2-2\beta}),\end{aligned}$$

which imply (3.33). We now need to estimate $|\mathbb{E}_{\ell_\alpha^k}(\mathcal{A} \circ F^{n-1} \cdot 1_{\tau \geq n-2}) - \Psi_{\alpha, n-1}(\eta_\alpha^k)|$ for all $\alpha \in \mathcal{A}$ and $k \in \mathbf{J}_\alpha$. Consider the case $n = 2$; we use Lemmata 3.7 and 3.9 which yield:

$$\begin{aligned}|\mathbb{E}_{\ell_\alpha^k}(\mathcal{A} \circ F) - \Psi_{\alpha, 1}(\eta_\alpha^k)| &\leq \\ &\leq C\|\mathcal{A}\|_1(\|r\|_{\ell_\alpha^k} \hat{y}^{-\beta} + Y'(\hat{y})\|\Delta h\|_{\ell_\alpha^k} + \|\Delta Y\|_{\ell_\alpha^k} \hat{y}^{-\beta}).\end{aligned}$$

The last term on the right hand side is $\mathcal{O}(\hat{y}^{-4\beta})$ by Lemma 3.15; in order to obtain (3.31) we are left with obtaining a bound for the following quantity:

$$(3.36) \quad \sum_{k \in \mathbf{J}_\alpha} c_\alpha^k C\|\mathcal{A}\|_1(\|r\|_{\ell_\alpha^k} \hat{y}^{-\beta} + Y'(\hat{y})\|\Delta h\|_{\ell_\alpha^k}).$$

Applying Lemma 3.1 gives a bound $\mathcal{O}(\hat{y}^{-2\beta} \log \hat{y})$ which gives (3.31) and completes the proof of (a) for $n = 2$.

Consider now the case $n \geq 3$; we assume by inductive hypothesis that Lemmata 3.12 and 3.11 hold for step $(n-1)$. We therefore obtain:

$$\begin{aligned}|\mathbb{E}_{\ell_\alpha^k}(\mathcal{A} \circ F^{n-1} \cdot 1_{\tau \geq n-2}) - \Psi_{\alpha, n-1}(\eta_\alpha^k)| &\leq \\ &\leq \|r\|_{\ell_\alpha^k} \mathbb{E}_{\ell_\alpha^k}(\mathcal{A} \circ F^{n-1} \cdot 1_{\tau \geq n-2}) + \|\mathcal{A}\|_1 C_n \hat{y}^{-2\beta} \log \hat{y}.\end{aligned}$$

Using once more Lemma 3.1 we estimate the sum $\sum_k c_\alpha^k \|r\|_{\ell_\alpha^k} = \mathcal{O}(\hat{y}^{-\beta})$; applying corollary 3.4 gives (3.31) and concludes the proof of (a) in the general case.

In order to prove part (b), it suffices to apply part (a) to both pairs; since ℓ_1 and ℓ_2 are close to each other, we can adjust N_α and \mathbf{J}_α of a bounded quantity in order to find a set of indices which is common to both ℓ_1 and ℓ_2 . In doing this we discard at most an uniformly bounded number of standard pairs ℓ_α^k , which contribute at most with $\|\mathcal{A}\| \mathcal{O}(\hat{y}^{-2\beta} \log \hat{y})$ and can therefore be neglected. Estimate (3.34) then follows by simple geometrical considerations similar to those used in the proof of Lemma 3.15. \square

We now conclude this section by proving Lemmata 3.11-3.13.

Proof of Lemma 3.11. Lemma 3.16b allows to write the following estimate:

$$\begin{aligned}|\Psi_{\alpha, k}(\eta_1) - \Psi_{\alpha, k}(\eta_2)| &\leq \\ &\leq \sum_{\alpha'} \sum_{j \in \mathbf{J}_{\alpha'}} |c_{\alpha', 1}^j \Psi_{\alpha', k-1}(\eta_{\alpha', 1}^j) - c_{\alpha', 2}^j \Psi_{\alpha', k-1}(\eta_{\alpha', 2}^j)| + \|\mathcal{A}\|_1 o(\hat{y}^{-1}).\end{aligned}$$

We now claim that, for any given α' we have:

$$(3.37) \quad \sum_{j \in \mathbf{J}_{\alpha'}} |c_{\alpha', 1}^j \Psi_{\alpha', k-1}(\eta_{\alpha', 1}^j) - c_{\alpha', 2}^j \Psi_{\alpha', k-1}(\eta_{\alpha', 2}^j)| = \|\mathcal{A}\|_1 o(\hat{y}^{-1}).$$

In order to estimate each term of the sum we write:

$$\begin{aligned}
 & |c_{\alpha',1}^j \Psi_{\alpha',k-1}(\eta_{\alpha',1}^j) - c_{\alpha',2}^j \Psi_{\alpha',k-1}(\eta_{\alpha',2}^j)| = \\
 (3.38a) \quad & = |(c_{\alpha',1}^j - c_{\alpha',2}^j) \Psi_{\alpha',k-1}(\eta_{\alpha',1}^j)| + \\
 (3.38b) \quad & + c_{\alpha',2}^j |\Psi_{\alpha',k-1}(\eta_{\alpha',1}^j) - \Psi_{\alpha',k-1}(\eta_{\alpha',2}^j)|.
 \end{aligned}$$

We obtain a bound for (3.38a) in the following way: define Θ_i as in the proof of Lemma 3.9 and, for fixed j and α' let ξ_i be the inverse function of Θ_i on $I_{\alpha'}$; then we can estimate:

$$\begin{aligned}
 |c_{\alpha',1}^j - c_{\alpha',2}^j| &= \bar{\rho}_\alpha \int_{I_{\alpha'}} \left| \frac{1}{\dot{\Theta}_1(\xi_1(\theta))} - \frac{1}{\dot{\Theta}_2(\xi_2(\theta))} \right| d\theta = \\
 &= \bar{\rho}_\alpha \int_{I_{\alpha'}} \frac{1}{\dot{\Theta}_1(\xi_1(\theta))} \left| 1 - \frac{\dot{\Theta}_1(\xi_1(\theta))}{\dot{\Theta}_2(\xi_2(\theta))} \right| d\theta = \\
 &\leq c_{\alpha',1}^j \left\| 1 - \frac{\dot{\Theta}_1(\xi_1(\theta))}{\dot{\Theta}_2(\xi_2(\theta))} \right\|_{\alpha'}^j.
 \end{aligned}$$

By simple geometrical considerations we obtain:

$$\left\| 1 - \frac{\dot{\Theta}_1(\xi_1)}{\dot{\Theta}_2(\xi_2)} \right\|_{\alpha'}^j \leq \left\| \frac{\ddot{\Theta}_2}{\dot{\Theta}_2^2} \right\|_{\alpha'}^j \cdot |Y(\eta_1) - Y(\eta_2)|.$$

from which we conclude, using once more Lemma 3.1 and corollary 3.4, that the sum over (3.38a) contributes with $o(\hat{y}^{-1})$.

In order to bound (3.38b), first assume that $k \geq 3$ and that we proved (3.16) at step $(k-1)$; we then apply estimate (3.34) and conclude by Lemma 3.11 at step $k-1$ that the contribution of (3.38b) is bounded by $o(\hat{y}^{-1})$ as well. For the base case $k=2$ we need to use Lemma 3.9, which yields the following bound for the contribution of (3.38b):

$$\sum_{j \in \mathbf{J}_\alpha} c_{\alpha',2}^j C \|\mathcal{A}\|_1 |Y(\eta_{\alpha,1}^j) - Y(\eta_{\alpha,2}^j)| \hat{y}^{-\beta}.$$

Using once again Lemma 3.1 and (3.34) we obtain a bound of order $o(\hat{y}^{-1})$, which concludes the proof. \square

Proof of Lemma 3.12. We apply Lemma 3.15 in order to obtain a reference pair $\bar{\ell}$; we claim that $\bar{\ell}$ satisfies (3.17). Define ℓ^* as we did in Lemma 3.7, i.e. $\ell^* = (\Gamma, \rho^*)$ where ρ^* is the uniform density on I . Then, as in Lemma 3.7, we need to estimate the quantity $|\mathbb{E}_\ell - \mathbb{E}_{\ell^*}| + |\mathbb{E}_{\ell^*} - \mathbb{E}_{\bar{\ell}}|$; by proposition 3.14 we can neglect the contribution of curves inside \mathbf{C}_1 , thus, by Lemma 3.16:

$$\begin{aligned}
 & |\mathbb{E}_\ell(\mathcal{A} \circ F^k \cdot 1_{\tau \geq k-1}) - \mathbb{E}_{\ell^*}(\mathcal{A} \circ F^k \cdot 1_{\tau \geq k-1})| = \\
 &= \sum_{\alpha \in \mathcal{A}} \sum_{j \in \mathbf{J}_\alpha} |c_\alpha^j - c_\alpha^{j*}| |\Psi_{\alpha,k-1}(\eta_\alpha^j)| + \|\mathcal{A}\| o(\hat{y}^{-1}).
 \end{aligned}$$

It is not difficult to check that

$$|c_\alpha^j - c_\alpha^{j*}| \leq C_\# \|r\| c_\alpha^{j*},$$

which implies that

$$\begin{aligned} |\mathbb{E}_\ell(\mathcal{A} \circ F^k \cdot 1_{\tau \geq k-1}) - \mathbb{E}_{\ell^*}(\mathcal{A} \circ F^k \cdot 1_{\tau \geq k-1})| &\leq \\ C_\# \|r\| |\mathbb{E}_{\ell^*}(\mathcal{A} \circ F^k \cdot 1_{\tau \geq k-1})| + \|\mathcal{A}\| o(\hat{y}^{-1}). \end{aligned}$$

We use once more Lemma 3.16b on ℓ^* and $\bar{\ell}$ noticing that, by construction, the η_α^j appearing in their respective (3.31) coincide; hence:

$$\begin{aligned} |\mathbb{E}_{\ell^*}(\mathcal{A} \circ F^k \cdot 1_{\tau \geq k-1}) - \mathbb{E}_{\bar{\ell}}(\mathcal{A} \circ F^k \cdot 1_{\tau \geq k-1})| &\leq \\ (3.39) \quad &\leq \sum_{\alpha \in \mathcal{A}} \sum_{j \in \mathbf{J}_\alpha} |c_\alpha^{j*} - \bar{c}_\alpha^j| |\Psi_{\alpha, k-1}(\eta_\alpha^j)| + \|\mathcal{A}\| o(\hat{y}^{-1}). \end{aligned}$$

We can estimate (3.39) in the following way:

$$\begin{aligned} |c_\alpha^{j*} - \bar{c}_\alpha^j| &\leq \int_{I_\alpha} \bar{\rho} \left| \frac{1}{\dot{\Theta}(x^*(\theta))} - \frac{1}{\dot{\Theta}(\bar{x}(\theta))} \right| d\theta \leq \\ &\leq c_\alpha^{j*} \left\| 1 - \frac{\dot{\Theta}(x^*(\theta))}{\dot{\Theta}(\bar{x}(\theta))} \right\|. \end{aligned}$$

By construction of $\bar{\ell}$, we know that $|\bar{x} - x^*| = \mathcal{O}(\hat{y}^{-3\beta})$, hence from the definition of standard curve, and from the fact that we consider points outside \mathbf{C}_1 we have:

$$\left\| 1 - \frac{\dot{\Theta}(x^*(\theta))}{\dot{\Theta}(\bar{x}(\theta))} \right\| \leq \mathcal{O}(\hat{y}^{-2\beta}),$$

which concludes the proof of (3.17). \square

Proof of Lemma 3.13. First of all we use Lemma 3.16 to obtain:

$$\Psi_{\alpha, k}(\eta) = \sum_{\alpha' \in \mathcal{A}} \sum_{j \in \mathbf{J}_{\alpha'}} c_{\alpha'}^j(\eta) \Psi_{\alpha', k-1}(\eta_{\alpha'}^j(\eta)) + \|\mathcal{A}\| o(\hat{y}^{-1});$$

for each $\alpha' \in \mathcal{A}$, let $\eta^*(\eta, \alpha, \alpha')$ such that $|Y(\eta^*) - Y(\eta)| < 1$ and $\pi_x F(\bar{x}_\alpha, \eta^*) = \bar{x}_{\alpha'}$. By (3.37) we conclude that:

$$\begin{aligned} \sum_{j \in \mathbf{J}_{\alpha'}} c_{\alpha'}^j(\eta) \Psi_{\alpha', k-1}(\eta_{\alpha'}^j(\eta)) &= \\ &= \sum_{j \in \mathbf{J}_{\alpha'}} c_{\alpha'}^j(\eta^*) \Psi_{\alpha', k-1}(\eta_{\alpha'}^j(\eta^*)) + \|\mathcal{A}\|_1 o(\hat{y}^{-1}). \end{aligned}$$

We claim that we can neglect the dependency of $c_{\alpha'}^j$ on η^* ; in fact, if η_1^*, η_2^* are such that $\pi_x(F(\bar{x}_\alpha, \eta_i^*)) = \bar{x}_{\alpha'}$, then necessarily $Y(\eta_1^*) \equiv Y(\eta_2^*) \pmod{1}$. Consequently, following the proof of Lemma 3.8 we obtain:

$$|c_{\alpha'}^j(\eta_1^*) - c_{\alpha'}^j(\eta_2^*)| \leq C |\eta_1^* - \eta_2^*| \hat{y}^{-1} \cdot c_{\alpha'}^j(\eta_1^*).$$

Since $\Psi_{\alpha', k-1} = \|\mathcal{A}\| \mathcal{O}(\hat{y}^{-\beta})$ by corollary 3.4, we conclude that we can find a sequence $c_{\alpha'}^j$ such that:

$$\sum_{j \in \mathbf{J}_{\alpha'}} c_{\alpha'}^j(\eta^*) \Psi_{\alpha', k-1}(\eta_{\alpha'}^j(\eta^*)) = \sum_{j \in \mathbf{J}_{\alpha'}} c_{\alpha'}^j \Psi_{\alpha', k-1}(\eta_{\alpha'}^j(\eta^*)) + \|\mathcal{A}\|_1 o(\hat{y}^{-1}),$$

that is:

$$\Psi_{\alpha,k}(\eta) = \sum_{j \in \mathbf{J}_{\alpha'}} c_{\alpha'}^j \Psi_{\alpha',k-1}(\eta_{\alpha'}^j(\eta^*)) + \|\mathcal{A}\|_1 o(\hat{y}^{-1}).$$

We now consider separately the cases $k = 2$ and $k \geq 3$; we first assume that $k = 2$: in this case, Lemma 3.8a implies that $\Psi_{\alpha',1} = \|\mathcal{A}\|_1 o(\hat{y}^{-1})$ for α' corresponding to curves which do not intersect the critical set $\hat{\mathbf{C}}_1$; we can therefore neglect the contribution of such curves. We let \mathcal{A}^* be the subset of \mathcal{A} given by indices associated to the remaining curves. We use Lemma 3.10 and (3.32) to obtain:

$$\begin{aligned} \sum_{j \in \mathbf{J}_{\alpha'}} c_{\alpha'}^j \Psi_{\alpha',k-1}(\eta_{\alpha'}^j(\eta^*)) &= \\ &= \sum_{j \in \mathbf{J}_{\alpha'}} c_{\alpha'}^j \sum_l' \hat{\Psi}_{\alpha',1}^{(l)} e^{-2\pi i l Y(\eta_{\alpha'}^j(\eta^*))} = \\ &= \sum_l \hat{\Psi}_{\alpha',1}^{(l)} e^{-2\pi i l Y(\eta_{\alpha'}^0(\eta^*))} \underbrace{\sum_{j \in \mathbf{J}_{\alpha'}} c_{\alpha'}^j e^{-2\pi i l (\mu_{\alpha'}(j - \theta_{\alpha'}^j + \theta_{\alpha'}^0) - \theta_{\alpha'}^j + \theta_{\alpha'}^0)}}_{\Upsilon_{\alpha'}^{(l)}}. \end{aligned}$$

We can now understand the cancellation mechanism: we claim that

$$(3.40) \quad |\Upsilon_{\alpha'}^{(l)}| \leq C \hat{y}^{1/2-\beta} + \min(C |l| \hat{y}^{1-2\beta} \log \hat{y} + C |l| \hat{y}^{-1}, 3);$$

In fact $\Upsilon_{\alpha'}^{(l)}$ is given by a sum of oscillating terms; the phase of each term differs from the phase of the previous one by $\mathcal{O}(\hat{y}^{-1})$, and we have $\mathcal{O}(\hat{y}^{2\beta})$ such terms. We will collect together the phases belonging to the same period: in an ideal (unrealistic) situation, standard pairs would have uniform weights, and therefore summing over each complete collection would give us a contribution of order \hat{y}^{-1} , by comparison with a Riemann sum. In reality standard pairs have non-uniform weights and we need more involved estimates in order to deal with the lack of uniformity of weights.

If $\tilde{h}_S(\bar{x}_\alpha) = 0$ it is necessary to avoid a portion of curve where $|\tilde{h}_1|$ is too small; namely, define $\Theta(x) = x + Y(\psi_{\alpha,\eta}(x))$ and let $x^*(\alpha, \alpha') \in I_\alpha$, such that $|\mu_{\alpha'}(\Theta(x^*)) - \mu_{\alpha'}(\Theta(\bar{x}_\alpha))| = 2$; let $h^* = |\tilde{h}_S(x^*)|$ and define $\mathcal{J}_{\alpha'} \subset \mathbf{J}_{\alpha'}$ such that $\forall j \in \mathcal{J}_{\alpha'}$ we have $|\tilde{h}(\theta_{\alpha'}^j)| \geq h^*$. It is easy to prove, by definition of $\mu_{\alpha'}^j$ and h , that $|\bar{x}_\alpha - x^*| \leq C_\# \cdot \hat{y}^{\frac{1}{2}-\beta}$ and $h \geq C_\# \hat{y}^{\frac{1}{2}-\beta}$, which immediately implies that $\sum_{j \in \mathbf{J}_{\alpha'} \setminus \mathcal{J}_{\alpha'}} c_{\alpha'}^j \leq C_\# \cdot \hat{y}^{\frac{1}{2}-\beta}$. On the other hand, if $\tilde{h}_S(\bar{x}_\alpha) \neq 0$ we can simply take $\mathcal{J}_{\alpha'} = \mathbf{J}_{\alpha'}$. Consider a partition $\mathcal{J}_{\alpha'}$ in subsets $\langle \mathcal{J}_{\alpha'} \rangle_i$ such that $j, j' \in \langle \mathcal{J}_{\alpha'} \rangle_i$ if and only if:

$$\lfloor \mu_{\alpha'}(j - \theta_{\alpha'}^j + \theta_{\alpha'}^0) \rfloor = \lfloor \mu_{\alpha'}(j' - \theta_{\alpha'}^{j'} + \theta_{\alpha'}^0) \rfloor.$$

Denote $\langle \dot{\Theta} \rangle_i = \min_{j \in \langle \mathcal{J}_{\alpha'} \rangle_i} |\tilde{h}_S(\theta_{\alpha'}^j) Y'(\hat{y})|$; (3.33) then implies:

$$(3.41) \quad \max_{j, j' \in \langle \mathcal{J}_{\alpha'} \rangle_i} (\theta_{\alpha'}^j - \theta_{\alpha'}^{j'}) \leq C_\# \cdot \hat{y} \langle \dot{\Theta} \rangle_i^{-1} = \mathcal{O}(\hat{y}^{\frac{1}{2}-\beta})$$

Define $\langle c_{\alpha'} \rangle_i = \sum_{j \in \langle \mathcal{J}_{\alpha'} \rangle_i} c_{\alpha'}^j$; notice that (3.41) implies

$$(3.42) \quad \langle c_{\alpha'}^j \rangle_i \leq C_\# \hat{y} \langle \dot{\Theta} \rangle_i^{-1},$$

and that, for any $j, j' \in \langle \mathcal{J}_{\alpha'} \rangle_i$:

$$(3.43) \quad \begin{aligned} & |e^{-2\pi i l \mu_{\alpha'}(j - \theta_{\alpha'}^j + \theta_{\alpha'}^0) - \theta_{\alpha'}^j + \theta_{\alpha'}^0} - e^{-2\pi i l \mu_{\alpha'}(j - \theta_{\alpha'}^{j'} + \theta_{\alpha'}^0) - \theta_{\alpha'}^{j'} + \theta_{\alpha'}^0}| \\ & \leq \min(C_{\#} |l| \hat{y} \langle \dot{\Theta} \rangle_i^{-1}, 2). \end{aligned}$$

We want to keep those $\langle \mathcal{J}_{\alpha'} \rangle_i$ for which $\{\mu_{\alpha'}(j - \theta_{\alpha'}^j + \theta_{\alpha'}^0) \bmod 1\}_{j \in \langle \mathcal{J}_{\alpha'} \rangle_i}$ samples \mathbb{T}^1 with an error bounded by $\mathcal{O}(\hat{y}^{-1})$. It is sufficient to discard the *first* and *last* (with respect to the natural ordering given by j) of the $\langle \mathcal{J}_{\alpha'} \rangle_i$; in fact, by (3.42), their contribution is bounded by $C_{\#} \hat{y}^{\frac{1}{2} - \beta}$. Define $\bar{j}_i = \min \langle \mathcal{J}_{\alpha'} \rangle_i$ and compute the following sum:

$$\sum_{j \in \langle \mathcal{J}_{\alpha'} \rangle_i} \frac{1}{|\langle \mathcal{J}_{\alpha'} \rangle_i|} e^{-2\pi i l \mu_{\alpha'}(j - \theta_{\alpha'}^{\bar{j}_i} + \theta_{\alpha'}^0)};$$

by (3.33), $\mu_{\alpha'}$ is almost a linear function in each $\langle \mathcal{J}_{\alpha'} \rangle_i$; in particular, $\forall j \in \langle \mathcal{J}_{\alpha'} \rangle_i$ the following bound holds true:

$$\mu_{\alpha'}(j - \theta_{\alpha'}^{\bar{j}_i} + \theta_{\alpha'}^0) = \mu_{\alpha'}(\bar{j}_i - \theta_{\alpha'}^{\bar{j}_i} + \theta_{\alpha'}^0) + \mu'_{\alpha'}(\bar{j}_i - \theta_{\alpha'}^{\bar{j}_i} + \theta_{\alpha'}^0)(j - \bar{j}_i) + \mathcal{O}(\hat{y}^{-1-2\beta}).$$

where $\mu'_{\alpha'}$ is the derivative of $\mu_{\alpha'}$. By hypothesis $\hat{\Psi}_{\alpha',1}^{(0)} = 0$, hence we can assume $l \neq 0$; comparison with the Riemann sum of $\int e^{2\pi i l \theta} d\theta$ implies:

$$(3.44) \quad \sum_{j \in \langle \mathcal{J}_{\alpha'} \rangle_i} \frac{1}{|\langle \mathcal{J}_{\alpha'} \rangle_i|} e^{-2\pi i l \mu_{\alpha'}(j - \theta_{\alpha'}^{\bar{j}_i} + \theta_{\alpha'}^0)} \leq \min(C_{\#} |l| \hat{y}^{-1}, 1).$$

Finally, by definition of $c_{\alpha'}^j$ we have the following estimate:

$$\left| \frac{c_{\alpha'}^j}{\langle c_{\alpha'} \rangle_i} - \frac{1}{|\langle \mathcal{J}_{\alpha'} \rangle_i|} \right| \leq C_{\#} \cdot \hat{y}^{2\beta} \langle \dot{\Theta} \rangle_i^{-2}.$$

We can therefore estimate $|\Upsilon_{\alpha'}^{(l)}|$ as follows:

$$(3.45a) \quad |\Upsilon_{\alpha'}^{(l)}| \leq \sum_i \langle c_{\alpha'} \rangle_i \sum_{j \in \langle \mathcal{J}_{\alpha'} \rangle_i} \left| \frac{c_{\alpha'}^j}{\langle c_{\alpha'} \rangle_i} - \frac{1}{|\langle \mathcal{J}_{\alpha'} \rangle_i|} \right| +$$

$$(3.45b) \quad \left| \sum_{j \in \langle \mathcal{J}_{\alpha'} \rangle_i} \frac{1}{|\langle \mathcal{J}_{\alpha'} \rangle_i|} (e^{-2\pi i l \mu_{\alpha'}(j - \theta_{\alpha'}^j + \theta_{\alpha'}^0) - \theta_{\alpha'}^j + \theta_{\alpha'}^0} + \right.$$

$$\left. - e^{-2\pi i l \mu_{\alpha'}(j - \theta_{\alpha'}^{\bar{j}_i} + \theta_{\alpha'}^0) - \theta_{\alpha'}^{\bar{j}_i} + \theta_{\alpha'}^0}) \right| +$$

$$(3.45c) \quad \left| \sum_{j \in \langle \mathcal{J}_{\alpha'} \rangle_i} \frac{1}{|\langle \mathcal{J}_{\alpha'} \rangle_i|} e^{-2\pi i l \mu_{\alpha'}(j - \theta_{\alpha'}^{\bar{j}_i} + \theta_{\alpha'}^0) - \theta_{\alpha'}^{\bar{j}_i} + \theta_{\alpha'}^0} \right|.$$

The sum in (3.45a) can be bounded using Lemma 3.1; consider the finite measure space whose elements are the subsets $\langle \mathcal{J}_{\alpha'} \rangle_i$ with measure $\langle c_{\alpha'} \rangle_i$; we let

$$f_i = \sum_{j \in \langle \mathcal{J}_{\alpha'} \rangle_i} \left| \frac{c_{\alpha'}^j}{\langle c_{\alpha'} \rangle_i} - \frac{1}{|\langle \mathcal{J}_{\alpha'} \rangle_i|} \right| \leq C_{\#} \cdot \hat{y}^{1+2\beta} \langle \dot{\Theta} \rangle_i^{-2}$$

and

$$\lambda = C_{\#} \cdot \hat{y}^{1-2\beta} \quad \alpha = 1/2.$$

This gives a bound $\mathcal{O}(\hat{y}^{\frac{1}{2}-\beta})$; the sum in (3.45b) can be bounded again by Lemma 3.1; this time let

$$X_i = \left| \sum_{j \in \langle \mathcal{J}_{\alpha'} \rangle_i} \frac{1}{|\langle \mathcal{J}_{\alpha'} \rangle_i|} (e^{-2\pi i l \mu_{\alpha'}(j - \theta_{\alpha'}^j + \theta_{\alpha'}^0) - \theta_{\alpha'}^j + \theta_{\alpha'}^0} + e^{-2\pi i l \mu_{\alpha'}(j - \bar{\theta}_{\alpha'}^j + \theta_{\alpha'}^0) - \bar{\theta}_{\alpha'}^j + \theta_{\alpha'}^0}) \right| \leq \min(C_{\#} |l| \hat{y} \langle \dot{\Theta} \rangle_i^{-1}, 2)$$

and $f_i = C_{\#} \hat{y}^{\beta-1/2} X_i$:

$$\lambda = C_{\#} \cdot \hat{y}^{\frac{1}{2}-\beta} \quad \alpha = 1.$$

which yields a bound of $\min(|l| C \hat{y}^{1-2\beta} \log \hat{y}, 2)$. The remaining term (3.45c) can be bounded directly using (3.44), which gives a bound of $\min(C_{\#} |l| \hat{y}^{-1}, 1)$ and concludes the proof of the estimate (3.40). Define

$$\hat{\Psi}_{\alpha,2}^{(\alpha',l)} = \hat{\Psi}_{\alpha',1}^{(l)} \Upsilon_{\alpha'}^{(l)} e^{2\pi i l \Phi_{\alpha'}},$$

where $\Phi_{\alpha'}$ is a phase to be fixed later. Notice that by (3.40) and since $\beta > \frac{1}{2}$ we obtain (3.19a). Summarizing we have:

$$\Psi_{\alpha,2}(\eta) = \sum_{\alpha' \in \mathcal{A}^*} \sum_l \hat{\Psi}_{\alpha,2}^{(\alpha',l)} e^{-2\pi i l (Y(\eta_{\alpha'}^0(\eta^*)) - \Phi_{\alpha'})} + \|\mathcal{A}\|_1 \mathcal{O}(\hat{y}^{-1}).$$

We now study the exponential term. By definition of η^* :

$$\eta_{\alpha'}^0(\eta^*) = \eta^* + 2\dot{\phi}(\bar{x}_{\alpha'}),$$

therefore we have

$$\begin{aligned} Y(\eta_{\alpha'}^0(\eta^*)) &= Y(\hat{y} + 2\dot{\phi}(\bar{x}_{\alpha'})) + (Y(\eta^*) - Y(\hat{y})) \\ &+ \int_{Y(\hat{y})}^{Y(\eta^*)} \left(\frac{Y'(y(Y) + 2\dot{\phi}(\bar{x}_{\alpha'}))}{Y'(y(Y))} - 1 \right) dY. \end{aligned}$$

By (3.33) we conclude that:

$$\frac{Y'(y(Y) + 2\dot{\phi}(\bar{x}_{\alpha'}))}{Y'(y(Y))} - 1 = \frac{Y'(\hat{y} + 2\dot{\phi}(\bar{x}_{\alpha'}))}{Y'(\hat{y})} - 1 + \mathcal{O}(\hat{y}^{-2});$$

define

$$\omega_{\alpha'} = \frac{Y'(\hat{y} + 2\dot{\phi}(\bar{x}_{\alpha'}))}{Y'(\hat{y})} - 1 = \mathcal{O}(\hat{y}^{-1});$$

notice that $\omega_{\alpha'}$ satisfies the bound (3.21) because of our choice of \mathcal{A}^* . Then by letting $\dot{\phi}_{\alpha'} = Y(\hat{y} + 2\dot{\phi}(\bar{x}_{\alpha'})) + Y(\eta^*) - Y(\hat{y}) - \omega_{\alpha'} \cdot Y(\hat{y})$ (recall that the fractional part of $Y(\eta^*)$ does not depend on η), we have:

$$|e^{-2\pi i l (Y(\eta_{\alpha'}^0(\eta^*)) - \Phi)} - e^{-2\pi i l \omega_{\alpha'} Y(\eta)}| \leq \min(C_{\#} |l| \hat{y}^{-2+2\beta}, 2).$$

By definition of $\hat{\Psi}_{\alpha,1}^{(l)}$ and $\hat{\Psi}_{\alpha,2}^{(l)}$ and by estimate (3.40) we then have:

$$\begin{aligned}
& \|\Psi_{\alpha,2}(\eta) - \sum_{\alpha^* \in \mathcal{A}^*} \sum_l \hat{\Psi}_{\alpha,2}^{(\alpha^*, l)} e^{-2\pi i l \omega_{\alpha^*} Y(\eta)}\| \leq \\
& \leq \sum_l |\mathcal{A}_l| \hat{y}^{-\beta} \cdot (C \hat{y}^{\frac{1}{2}-\beta} + \min(|l| C \hat{y}^{1-2\beta} \log \hat{y} + |l| C \hat{y}^{-1}, 3)) \cdot \\
& \quad \cdot \min(C_{\#} |l| \hat{y}^{-2+2\beta}, 2) \leq \\
& \leq \sum_l |l|^2 |\mathcal{A}_l| o(\hat{y}^{-1}).
\end{aligned}$$

Assume now $k \geq 3$; define $Y_{j,\alpha'}(\eta^*) = Y(\eta_{\alpha'}^0(\eta^*)) + j + \mu_{\alpha'}(j)$; by Lemma 3.16 we have $|Y_{j,\alpha'} - Y(\eta_{\alpha'}^j(\eta^*))| < 1$, so that first by Lemma 3.11 and then by inductive hypothesis we can write:

$$\begin{aligned}
& \sum_{j \in \mathbf{J}_{\alpha'}} c_{\alpha'}^j \Psi_{\alpha',k-1}(\eta_{\alpha'}^j(\eta^*)) = \\
& = \sum_{j \in \mathbf{J}_{\alpha'}} c_{\alpha'}^j \Psi_{\alpha',k-1}(\eta(Y_j)) + \|\mathcal{A}\| o(\hat{y}^{-1}) = \\
& = \sum_{j \in \mathbf{J}_{\alpha'}} c_{\alpha'}^j \sum_{\alpha^*, l} \hat{\Psi}_{\alpha',k-1}^{(\alpha^*, l)} e^{2\pi i l \omega_{\alpha^*} Y_j} + \|\mathcal{A}\| o(\hat{y}^{-1}) = \\
& = \sum_{\alpha^*, l} \hat{\Psi}_{\alpha',k-1}^{(\alpha^*, l)} e^{-2\pi i l \omega_{\alpha^*} Y(\eta_{\alpha'}^0)} \underbrace{\sum_{j \in \mathbf{J}_{\alpha'}} c_{\alpha'}^j e^{-2\pi i l \omega_{\alpha^*} (j + \mu_{\alpha'}(j))}}_{\Upsilon_{\alpha^*, \alpha'}^{(l)}} + \|\mathcal{A}\| o(\hat{y}^{-1}).
\end{aligned}$$

We claim that

$$(3.46) \quad |\Upsilon_{\alpha^*, \alpha'}^{(l)}| \leq C \hat{y}^{\frac{1}{2}-\beta} + \min(C |l| \hat{y}^{-1}, 1).$$

Define in the same way as for the case $k = 2$ the index set $\mathcal{J}_{\alpha'} \subset \mathbf{J}_{\alpha'}$ and fix a partition in subsets $\langle \mathcal{J}_{\alpha'} \rangle_i$ such that $j, j' \in \langle \mathcal{J}_{\alpha'} \rangle_i$ if and only if:

$$[\omega_{\alpha^*} \cdot (j + \mu_{\alpha'}(j))] = [\omega_{\alpha^*} \cdot (j' + \mu_{\alpha'}(j'))].$$

Define $\langle \dot{\Theta} \rangle_i$, $\langle c_{\alpha'} \rangle_i$ and \bar{j}_i as before. Notice that (3.41) still holds true, hence so do all estimates on $\langle c_{\alpha'} \rangle_i$ and $\langle \dot{\Theta} \rangle_i$. We discard once more the first and last sets $\langle \mathcal{J}_{\alpha'} \rangle_i$; for the remaining ones the following estimate holds true:

$$\omega_{\alpha^*}(j + \mu_{\alpha'}(j)) = \omega_{\alpha^*}(\bar{j}_i + \mu_{\alpha'}(\bar{j}_i)) + \omega_{\alpha^*}(j - \bar{j}_i + \mathcal{O}(1)),$$

hence, by comparison with a Riemann sum of $\int e^{2\pi i l \theta} d\theta$ we obtain the following estimate:

$$(3.47) \quad \sum_j \frac{1}{|\langle \mathcal{J}_{\alpha'} \rangle_i|} e^{-2\pi i l \mu_{\alpha^*}(j + \mu_{\alpha'}(j))} \leq \min(C |l| \hat{y}^{-1}, 1).$$

So that now we can estimate:

$$(3.48a) \quad |\Upsilon_{\alpha^*, \alpha'}^{(l)}| \leq \sum_i \langle c_{\alpha'} \rangle_i \sum_{j \in \langle \mathcal{J}_{\alpha'} \rangle_i} \left| \frac{c_{\alpha'}^j}{\langle c_{\alpha'} \rangle_i} - \frac{1}{|\langle \mathcal{J}_{\alpha'} \rangle_i|} \right| +$$

$$(3.48b) \quad + \left| \sum_{j \in \langle \mathcal{J}_{\alpha'} \rangle_i} \frac{1}{|\langle \mathcal{J}_{\alpha'} \rangle_i|} e^{-2\pi i l \omega_{\alpha^*} \cdot (j + \mu_{\alpha'}(j))} \right|.$$

The estimate for (3.48a) is the same as for (3.45a); the estimate for (3.48b) is given by (3.47) from which we conclude the proof of (3.46). Define

$$\hat{\Psi}_{\alpha, k}^{(\alpha^*, l, \alpha')} = \hat{\Psi}_{\alpha', k-1}^{(\alpha^*, l)} \Upsilon_{\alpha^*, \alpha'}^{(l)} e^{2\pi i l \Phi_{\alpha'}}$$

where $\Phi_{\alpha'}$ is a phase to be determined later. Hence we can write:

$$\Psi_{\alpha, k}(\eta) = \sum_{\alpha^* \in \mathcal{A}} \sum_{\alpha' \in \mathcal{A}} \sum_{l \in \mathbb{Z}} \hat{\Psi}_{\alpha, k}^{(\alpha^*, l, \alpha')} e^{-2\pi i l \omega_{\alpha^*} (Y(\eta_{\alpha'}^0(\eta^*)) - \Phi_{\alpha'})}$$

We now need to study the oscillating term $e^{-2\pi i l \omega_{\alpha^*} Y(\eta_{\alpha'}^0(\eta^*))}$; we now let $\Phi_{\alpha'} = Y(\hat{y} + 2\dot{\phi}(\bar{x}_{\alpha'})) - Y(\hat{y}) - \omega'_{\alpha} \cdot Y(\hat{y})$ so that $Y(\eta_{\alpha'}^0(\eta^*)) = \Phi_{\alpha'} + Y(\eta^*) + \mathcal{O}(\hat{y}^{-1+2\beta})$, which implies:

$$e^{-2\pi i l \omega_{\alpha^*} Y(\eta_{\alpha'}^0(\eta^*))} = e^{-2\pi i l \omega_{\alpha^*} Y(\eta^*)} + \min(|l| \mathcal{O}(\hat{y}^{-2+2\beta}), 1),$$

whose main term does not depend on α' . We can thus define $\hat{\Psi}_{\alpha, k}^{(\alpha^*, l)} = \sum_{\alpha' \in \mathcal{A}} \hat{\Psi}_{\alpha, k}^{(\alpha^*, l, \alpha')}$ which, by (3.46), implies (3.19b) and therefore (3.20). We then have:

$$\begin{aligned} \|\Psi_{\alpha, k}(\eta) - \sum_{\alpha^* \in \mathcal{A}^*} \sum_l \hat{\Psi}_{\alpha, k}^{(\alpha^*, l)} e^{-2\pi i l \omega_{\alpha^*} Y(\eta)}\| &\leq \\ &\leq \sum_l |l| |\hat{\mathcal{A}}| (C \hat{y}^{-\beta - (k-1)(\beta - \frac{1}{2})} \log \hat{y} + \min(|l| \hat{y}^{-1}, 1)) \cdot \\ &\quad \cdot \min(C_{\#} |l| \hat{y}^{-2+2\beta}, 2) \leq \\ &\leq \sum_l |l|^2 |\hat{\mathcal{A}}| o(\hat{y}^{-1}) \end{aligned}$$

□

Remark 3.17. *The proof of Lemma 3.13 clarifies why the cancellation argument fails for $\gamma \leq 2$; consider for instance the case $\gamma = 2$, then, in the formula for Υ we have a sum of $\mathcal{O}(\hat{y})$ terms whose phases differ by $\mathcal{O}(\hat{y}^{-1})$; hence, incomplete boundary collections of indices will contribute with some fixed proportion, and we cannot expect to iterate efficiently the cancellation scheme.*

As we mentioned in the introductory section, we performed simple numerical computations for the functions $\Psi_{\alpha, k}$ for various values of γ ; if γ is far enough away from 2, the outcome of such computations well agree with (3.10) apart from the $o(\hat{y}^{-1})$ term, which appears to be a technical byproduct of our techniques and, in principle, could be avoided by improving some of the estimates. As $\gamma \rightarrow 2^+$, the asymptotic behavior (3.10) appears to

dominate only for larger and larger values of \hat{y} ; it is thus increasingly delicate to obtain sensible quantitative results in this region, nevertheless the asymptotics (3.10) still appears to be the best fit.

4. COMPARISON WITH A BIASED RANDOM WALK

In this section we describe a procedure which allows to compare the dynamics on a standard pair with a one-dimensional biased random walk; the comparison argument is the crucial ingredient for the proof of Lemma 2.13. All arguments given in this section are adapted from the analogous ones explained in [3]; the only possibly non-trivial adaptation is the proof of proposition 4.3.

Let us denote by $\bar{\ell}$ the standard pair appearing in the statement of Lemma 2.13: we will call $\bar{\ell}$ the *master* standard pair. For $k \in \mathbb{Z}$ define $R_k = 2^k \hat{y}_{\bar{\ell}}$; we say that a standard pair ℓ is *close* to R_k if

$$\Gamma_\ell \subset \mathbb{T}^1 \times [R_k - 2A\nu, R_k + 2A\nu],$$

where ν is the one given by Lemma 3.6. We say that a standard pair ℓ is *compatible* with R_k if

$$\Gamma_\ell \subset \mathbb{T}^1 \times [R_{k-1}, R_{k+1}].$$

Definition 4.1. Let ℓ be a standard pair; following definition 2.12 we introduce the function

$$\tau_\ell^{[k]} : \Gamma_\ell \rightarrow \mathbb{N} \cup \{\infty\}$$

given by the following recursive definition: if ℓ is not compatible with R_k we let $\tau_\ell^{[k]} \equiv 0$. Otherwise let $p \in \Gamma_\ell$, then by item (b) of lemma 2.11 we have three possibilities:

- Fp belongs to a standard pair $\ell' = (\Gamma_{\ell'}, \rho_{\ell'})$: we then let $\tau_\ell^{[k]}(p) = \tau_{\ell'}^{[k]}(Fp) + 1$;
- Fp belongs to a stand-by pair, hence F^2p belongs to a standard pair ℓ'' : we then let $\tau_\ell^{[k]}(p) = \tau_{\ell''}^{[k]}(F^2p) + 2$;
- otherwise we let $\tau_\ell^{[k]}(p) = 0$.

Notice that by definition we have $\tau_\ell^{[k]} \leq \tau_\ell$.

Definition 4.2. Let ℓ be a standard pair close to R_k ; we then define a function $\xi_\ell^{[k]} : \Gamma_\ell \rightarrow \{-1, +1\}$ in the following way:

$$\xi_\ell^{[k]}(p) = \begin{cases} +1 & \text{if } \tau_\ell^{[k]}(p) < \tau_\ell(p) \text{ and } F^{\tau_\ell^{[k]}(p)}(p) \text{ belongs to } \ell' \text{ close to } R_{k+1}; \\ -1 & \text{otherwise.} \end{cases}$$

The main technical result of this section, which will be used to prove lemma 2.13 is given by the following

Proposition 4.3. Let ℓ be a standard pair; then if $\gamma > 2$:

- (a) there exists $0 < \vartheta < 1$ such that $\mathbb{P}_\ell(\tau_\ell^{[k]} \geq s) \leq C_\# \vartheta^{s \hat{y}_\ell^{-2}}$;
- (b) if ℓ is close to R_k , then we have $\mathbb{P}_\ell(\xi_\ell^{[k]} = -1) \geq 0.6$;

We now show how proposition 4.3 implies lemma 2.13 and postpone its proof to the end of the current section. Define two sequence of functions on the master standard pair $\bar{\ell}$:

$$\tau_k : \Gamma_{\bar{\ell}} \rightarrow \mathbb{N} \cup \{\infty\} \quad \chi_k : \Gamma_{\bar{\ell}} \rightarrow \mathbb{Z},$$

such that if $\tau_k(p) < \tau(p)$, then $F^{\tau_k(p)}p$ belongs to a standard pair ℓ' which is close to $R_{\chi_n(p)}$. We proceed by induction: let $\tau_0 \equiv 0$ and $\chi_0 \equiv 0$; assume we already defined τ_k and χ_k : then if $\tau_k(p) = \tau(p)$ we set:

$$\tau_{k+1}(p) \doteq \tau_k(p) \quad \chi_{k+1}(p) \doteq \chi_k(p) - 1.$$

Otherwise, by definition $F^{\tau_k(p)}(p) = p'$ belongs to some standard pair ℓ' close to $R_{\chi_k(p)}$; we then define:

$$\tau_{k+1}(p) \doteq \tau_k(p) + \tau_{\ell'}^{[\chi_k(p)]}(p') \quad \chi_{k+1}(p) \doteq \chi_k(p) + \xi_{\ell'}^{[\chi_k(p)]}(p').$$

The proof of lemma 2.13 now follows from the same argument which has been used in [3] to prove the corresponding estimate (23); we sketch the argument here and refer the reader to the said reference for the detailed proofs, which could be repeated verbatim in our situation. The crucial observation is that item (b) of proposition 4.3 implies that we can compare the dynamics of our system outside \mathbf{C}_2 with a biased random walk moving up with probability 0.4 and moving down with probability 0.6; by this comparison, and by item (a) of proposition 4.3 we obtain that, almost every point on a standard pair will visit the \mathbf{C}_2 (which includes the region $\{y \leq y_*\}$) in finite time, that is the statement of lemma 2.13.

We are now left concluding with the

Proof of proposition 4.3. First of all notice that if ℓ is not compatible with R_k , then item (a) is trivially satisfied; we therefore assume that ℓ is compatible with R_k . Define the following function on ℓ :

$$\zeta_n(x_0) \doteq \dot{\phi}(x_{n+\nu+1}) \mathbf{1}_{\tau_{\ell}^{[k]} \geq n} \mathbf{1}_{\tau_{\ell} \geq n+\nu},$$

We claim that the following expressions hold

$$(4.1a) \quad \mathbb{E}_{\ell}(\zeta_n) = \mathbb{P}_{\ell}(\tau_{\ell}^{[k]} \geq n) o(\hat{y}_{\ell}^{-1})$$

$$(4.1b) \quad \mathbb{E}_{\ell} \left(\left(\sum_{n=1}^N \zeta_n \right)^2 \right) \geq N \cdot 2A^2 \mathbb{P}_{\ell}(\tau_{\ell}^{[k]} \geq N) + o(N).$$

We now show that equations (4.1) imply proposition 4.3: in fact by definition of ζ_n we have:

$$\left\| \sum_{n=0}^N \zeta_n \right\| < 3\hat{y}_{\ell} + 2A(\nu + 1),$$

hence

$$\mathbb{E}_{\ell} \left(\left(\sum_{n=1}^N \zeta_n \right)^2 \right) \leq 9\hat{y}_{\ell}^2 + o(\hat{y}_{\ell}^2)$$

which, by (4.1b) implies:

$$N \cdot 2A^2 \mathbb{P}_{\ell}(\tau_{\ell}^{[k]} \geq N) \leq 9\hat{y}_{\ell}^2 + o(\hat{y}_{\ell}^2).$$

Taking $N = L\hat{y}_\ell^2$ for large enough L and dividing the previous expression by $2A^2L\hat{y}_\ell^2$ we thus obtain:

$$(4.2) \quad \mathbb{P}_\ell(\tau_\ell^{[k]} \geq L\hat{y}_\ell^2) \leq \vartheta$$

for some $0 < \vartheta < 1$. The previous expression implies item (a) by the following argument: for $m \in \mathbb{N}$ let

$$\mathcal{N}_m = [\lceil m \cdot L\hat{y}_\ell^2 \rceil, \lfloor (m+1) \cdot L\hat{y}_\ell^2 \rfloor].$$

thus $s \in \mathcal{N}_m$ for some $m \in \mathbb{N}$. Consequently, for any $m' \leq m$ we have that $F^{m'L\hat{y}_\ell^2}\{\tau_\ell^{[k]} \geq s\}$ can be decomposed in standard pairs and on each one we can apply (4.2) since by definition such standard pairs are still compatible with R_k . Hence, by induction we obtain

$$\mathbb{P}_\ell(\tau_\ell^{[k]} \geq s) \leq \vartheta^m$$

which implies item (a). To prove item (b), use (4.1a) and write:

$$\begin{aligned} \mathbb{E}_\ell \left(\sum_{n=1}^{\infty} \zeta_n \right) &\leq \sum_n \mathbb{P}_\ell \left(\tau_\ell^{[k]} \geq n \right) o(\hat{y}^{-1}) \\ &\leq \mathbb{E}_\ell \left(\tau_\ell^{[k]} \right) o(\hat{y}^{-1}). \end{aligned}$$

By item (a) we know that $\mathbb{E}_\ell \left(\tau_\ell^{[k]} \right) = \mathcal{O}(\hat{y}_\ell^2)$; on the other hand:

$$\mathbb{E}_\ell \left(\sum_{n=1}^{\infty} \zeta_n \right) = \hat{y} \cdot \mathbb{P}_\ell \left(\xi_\ell^{[k]} = +1 \right) + \lambda \hat{y} \cdot \mathbb{P}_\ell \left(\xi_\ell^{[k]} = -1 \right) + \mathcal{O}(1),$$

where $\lambda \in (-1/2, 1)$; dividing by \hat{y} we obtain:

$$\mathbb{P}_\ell \left(\xi_\ell^{[k]} = +1 \right) + \lambda \mathbb{P}_\ell \left(\xi_\ell^{[k]} = -1 \right) = o(1)$$

which implies:

$$\mathbb{P}_\ell \left(\xi_\ell^{[k]} = -1 \right) = \frac{1}{1-\lambda} (1 + o(1)) > 0.6$$

that is item (b).

We now only need to prove equations (4.1): By applying n times lemma 2.11 and discarding those pairs that do not satisfy $\tau_\ell^{[k]} \geq n$ we obtain the following decomposition:

$$(4.3) \quad F^n \ell = \bigcup_j \ell'_j \cup \bigcup_n \tilde{\ell}_n \cup \{\tau_\ell^{[k]} < n\}$$

and

$$F \bigcup_l \tilde{\ell}_l = \bigcup_j \ell''_j.$$

where ℓ'_j and ℓ''_j are standard pairs. Let $c'_j = \mathbb{P}_\ell(F^{-n}\Gamma_{\ell'_j})$ and $c''_j = \mathbb{P}_\ell(F^{-n-1}\Gamma_{\ell''_j})$; then, by definition we have

$$\sum_j c'_j + \sum_j c''_j = \mathbb{P}_\ell(\tau_\ell^{[k]} \geq n).$$

Thus, using lemma 3.6 we have:

$$\begin{aligned}\mathbb{E}_\ell(\zeta_n) &= \sum_j c'_j \mathbb{E}_{\ell'_j} \left(\dot{\phi} \circ F^{\nu+1} 1_{\tau_\ell \geq \nu} \right) + \sum_j c''_j \mathbb{E}_{\ell''_j} \left(\dot{\phi} \circ F^\nu 1_{\tau_\ell \geq \nu-1} \right) \\ &\leq \sum_j c'_j o(\hat{y}_\ell^{-1}) + \sum_j c''_j o(\hat{y}_\ell^{-1})\end{aligned}$$

which implies (4.1a). We now apply the same argument to the functions ζ_n^2 and obtain:

$$\mathbb{E}_\ell(\zeta_n^2) = \sum_j c'_j \mathbb{E}_{\ell'_j} \left(\dot{\phi}^2 \circ F^{\nu+1} 1_{\tau_\ell \geq \nu} \right) + \sum_j c''_j \mathbb{E}_{\ell''_j} \left(\dot{\phi}^2 \circ F^\nu 1_{\tau_\ell \geq \nu-1} \right);$$

whence, extracting the average value $\dot{\phi}^2 = (\dot{\phi}^2 - 2A^2) + 2A^2$ we obtain:

$$(4.4) \quad \mathbb{E}_\ell(\zeta_n^2) = (2A^2 + \mathcal{O}(\hat{y}_\ell^{-\beta})) \mathbb{P}_\ell(\tau_\ell^{[k]} \geq n) + o(\hat{y}_\ell^{-1})$$

Next, we claim that, for $m \in \mathbb{N}$ we have:

$$(4.5) \quad \mathbb{E}_\ell \left(\zeta_m \sum_{i=0}^{m-1} \zeta_i \right) = o(1).$$

In fact, applying decomposition (4.3) to F^m , we obtain:

$$\begin{aligned}\mathbb{E}_\ell \left(\zeta_m \sum_{i=0}^{m-1} \zeta_i \right) &= \sum_j c'_j \mathbb{E}_{\ell'_j} \left(\zeta_0 \sum_{i=-m}^{-1} \zeta_i \right) + \\ &\quad + \sum_j c''_j \mathbb{E}_{\ell''_j} \left(\zeta_{-1} \sum_{i=-m-1}^{-2} \zeta_i \right).\end{aligned}$$

We estimate separately the contribution of each standard pair in each of the two terms on the right hand side: fix ℓ'_j and define on $\Gamma_{\ell'_j}$ the function $B = \sum_{i=-m}^{-1} \zeta_i$; we want to prove that $\mathbb{E}_\ell(B\zeta_0) = o(1)$. Fix $p > \nu + 1$ to be determined later and decompose $B = B_1 + B_2$ as follows:

$$B_1 = \sum_{i=-m}^{-p} \zeta_i \quad \quad B_2 = \sum_{i=-p+1}^{-1} \zeta_i.$$

where if $m < p$ we assume conventionally that $B_1 = 0$ and $B_2 = B$. By definition of $\tau_\ell^{[k]}$ we have $\|B_1\|_\infty \leq 3\hat{y} + 2(\nu + 1)A \leq 4\hat{y}$; moreover B_1 depends only on x_i with $i < -(p - \nu - 1)$, hence, by item (a) of lemma 2.11:

$$\|\dot{B}_1\|_\infty = \mathcal{O}(\hat{y}^{-(p-\nu-1)\beta})$$

To estimate the contribution of B_1 , we write $B_1 = \bar{B}_1 + \tilde{B}_1$, where \bar{B}_1 is the constant part of B_1 ; then $\|\tilde{B}_1\|_\infty = \mathcal{O}(\hat{y}^{-(p-\nu-1)\beta})$ and we can write, using corollary 3.4 and theorem 3.6 and requiring p to be large enough, that:

$$\mathbb{E}_{\ell'_j}(B_1\zeta_0) = o(1).$$

Consider now the remaining term B_2 ; by definition we have $\|B_2\|_\infty \leq 2A(p-1)$; moreover, if x_ν belongs to a standard pair we have:

$$\left\| \frac{dB_2}{dx_\nu} \right\| = \mathcal{O}(1),$$

so that we obtain by corollary 3.4:

$$\mathbb{E}_{\ell'_j}(B_2\zeta_0) = o(\hat{y}^{-1}).$$

The terms involving ℓ''_j can be treated analogously and, by linearity of the expectation, we can conclude that 4.5 holds. Finally, using (4.4) and (4.5) we obtain:

$$\begin{aligned} \mathbb{E}_\ell \left(\left(\sum_{i=0}^N \zeta_i \right)^2 \right) &= \sum_{i=0}^N (2A^2 \mathbb{P}_\ell(\tau_\ell \geq i) + o(1)) \\ &\geq N \cdot 2A^2 \mathbb{P}_\ell(\tau_\ell \geq N) + N \cdot o(1). \end{aligned}$$

which concludes the proof. \square

APPENDIX A. PROOF OF LEMMA 2.9

In order to prove items (a_1) and (a_2) , it suffices to check that, for any standard curve Γ , the estimates

$$(A.1) \quad \left. \frac{d}{dx} \tilde{h}_1 \right|_\Gamma = \mathcal{O}(1) \quad \left. \frac{d}{dx} \tilde{h}_1(\tilde{h}_1 \circ F) \right|_\Gamma = \mathcal{O}(1)$$

hold in the specified neighborhood; then the proof trivially follows from the definition of $\hat{\mathbf{C}}_1$ and $\hat{\mathbf{C}}_2$. In turn (A.1) easily follows from the definition of \tilde{h}_1 ; in fact recall that

$$\partial_x \tilde{h}_1 = \mathcal{O}(1) \quad \partial_y \tilde{h}_1 = \mathcal{O}(y^{-1});$$

thus

$$\left. \frac{d}{dx} \tilde{h}_1 \right|_\Gamma = \partial_x \tilde{h}_1 + Y' h_\Gamma \cdot \partial_y \tilde{h}_1 = \mathcal{O}(1);$$

which concludes the proof of item (a_1) . Similarly, notice that:

$$\left. \frac{d}{dx} \tilde{h}_1(\tilde{h}_1 \circ F) \right|_\Gamma = \left. \frac{d}{dx} \tilde{h}_1 \right|_\Gamma \cdot \tilde{h}_1 \circ F + \tilde{h}_1 \cdot \left. \frac{d}{dx} \tilde{h}_1 \circ F \right|_\Gamma;$$

the first term of the right hand side is $\mathcal{O}(1)$ by the previous argument, on the other hand the second term can be bounded as follows:

$$\tilde{h}_1 \cdot \left. \frac{d}{dx} \tilde{h}_1 \circ F \right|_\Gamma = \tilde{h}_1 \cdot \left(\partial_x \tilde{h}_1 \left. \frac{dx_1}{dx} \right|_\Gamma + \partial_y \tilde{h}_1 \left. \frac{dy_1}{dx} \right|_\Gamma \right).$$

By definition $\left. \frac{dx_1}{dx} \right|_\Gamma = Y' \tilde{h}_\Gamma = \mathcal{O}(y^\beta)$, which implies that the first term is $\mathcal{O}(1)$; moreover, by (2.9) we have that $\left. \frac{dy_1}{dx} \right|_\Gamma = \mathcal{O}(Y') \left. \frac{dx_1}{dx} \right|_\Gamma$, from which we conclude the proof of item (a_2) . The proof of item (b_1) is simple, since, we have that $\left. \frac{d}{dx} \tilde{h}_1 \right|_\Gamma$ is bounded away from zero in $\Gamma \cap \hat{\mathbf{C}}_1$. Concerning item (b_2) , it is not difficult to see, by direct inspection (see Figure 1) that the number of connected components of $\Gamma \cap \hat{\mathbf{C}}_2$ is bounded by $N_\Gamma + 4$, where N_Γ is the number of intersections of $F(\Gamma \cap \hat{\mathbf{C}}_1)$ with the vertical line $\{x = 0\}$; it is thus sufficient to prove that the number of such intersections is uniformly bounded in \hat{y}_Γ . This, however, is simple to achieve since, by definition, Γ has a quadratic critical point inside $\hat{\mathbf{C}}_1$, and its curvature is bounded from above by $4AY'(\hat{y}_\Gamma)$, therefore its image will intersect the said vertical line

in at most $2 \cdot 4AY'(\hat{y}_\Gamma)\hat{K}_1^2\hat{y}_\Gamma^{-2\beta} = \mathcal{O}(1)$ points, which proves item (b_2) . To prove item (c_1) , notice that, by definition:

$$|\tilde{h}_1(x_0, y_0)| = |2\ddot{\phi}(x_0) + 1/Y'(y_{-1}) + 1/Y'(y_0)|;$$

using (2.4) we can write:

$$\begin{aligned} \hat{\mathbf{C}}_1 &\subset \{(x, y) \text{ s.t. } |2\ddot{\phi}(x)| < 2K_1Y'(y)^{-1/2}\} \\ &\subset \{|x| < \text{Const} \cdot Y'(y)^{-1/2}\} \cup \{|x - 1/2| < \text{Const} \cdot Y'(y)^{-1/2}\}. \end{aligned}$$

Denote the two sets that appear in the last expression by $\mathbf{C}_1^{(0)}$ and $\mathbf{C}_1^{(1)}$ respectively; the Lebesgue measure of $\mathbf{C}_1^{(i)}$ is finite if the function $Y'^{-1/2}$ is integrable at ∞ , i.e. if $\beta > 1$, that is, if $\gamma > 3$. In the same way we can obtain a lower bound, so that if $\gamma \leq 3$ then $\text{Leb}(\mathbf{C}_1) = \infty$. Similarly, in order to prove item (c_2) , define, for $i \in \{0, 1\}$ and $n \in \mathbb{N}$:

$$\hat{\mathbf{C}}_2^{(i,n)} = \hat{\mathbf{C}}_2 \cap \hat{\mathbf{C}}_1^{(i)} \cap \{(x, y) \text{ s.t. } x + Y(y) \in [n/2, (n+1)/2]\};$$

also let $\hat{y}_n = \inf_{(x,y) \in \hat{\mathbf{C}}_2^{(i,n)}} y \sim n^{1/\gamma}$. Then, for each $\hat{\mathbf{C}}_2^{(i,n)}$, consider the following decomposition (see also figure 2):

$$\begin{aligned} \hat{\mathbf{C}}_2'^{(i,n)} &= \{(x_0, y_0) \in \hat{\mathbf{C}}_2^{(i,n)} \text{ s.t. } |\tilde{h}_1(x_1, y_1)| < (\hat{K}_2/\hat{K}_1)Y'(y_1)^{-1/2}\} \\ \hat{\mathbf{C}}_2''^{(i,n)} &= \{(x_0, y_0) \in \hat{\mathbf{C}}_2^{(i,n)} \text{ s.t. } |\tilde{h}_1(x_1, y_1)| < A\} \setminus \hat{\mathbf{C}}_2'^{(i,n)} \\ \hat{\mathbf{C}}_2'''^{(i,n)} &= \hat{\mathbf{C}}_2^{(i,n)} \setminus (\hat{\mathbf{C}}_2'^{(i,n)} \cup \hat{\mathbf{C}}_2''^{(i,n)}). \end{aligned}$$

First consider $(x, y) \in \hat{\mathbf{C}}_2'''^{(i,n)}$; by definition we have:

$$|\tilde{h}_1(x, y)| < \frac{2\hat{K}_2}{AY'(y)}$$

which is a bound for x of order $\mathcal{O}(y^{-2\beta})$, so that:

$$\text{Leb}(\hat{\mathbf{C}}_2'''^{(i,n)}) \leq C_{\#}\hat{y}_n^{-4\beta}.$$

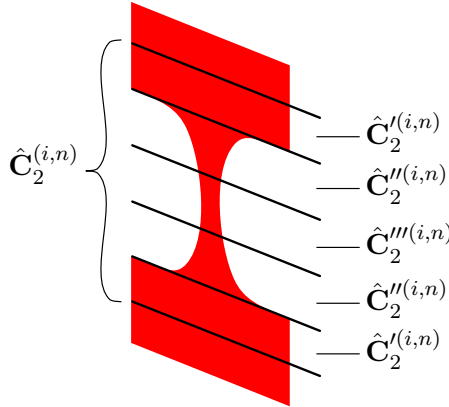


FIGURE 2. Decomposition of $\hat{\mathbf{C}}_2^{(i,n)} = \hat{\mathbf{C}}_2'^{(i,n)} \cup \hat{\mathbf{C}}_2''^{(i,n)} \cup \hat{\mathbf{C}}_2'''^{(i,n)}$.

The measure of $\hat{\mathbf{C}}_2'^{(i,n)}$ and $\hat{\mathbf{C}}_2''^{(i,n)}$ can be estimated using the following change of variables:

$$(x_0, y_0) \mapsto (\xi, \eta) = (\tilde{h}_1(x_0, y_0), \tilde{h}_1(x_1, y_1));$$

this map is an invertible diffeomorphism and its Jacobian determinant is of order $Y'(\hat{y}_n)$; for convenience denote $Y'_n = Y'(\hat{y}_n)$. Therefore, for $\hat{\mathbf{C}}_2'^{(i,n)}$ we obtain:

$$\text{Leb}(\hat{\mathbf{C}}_2'^{(i,n)}) \leq \frac{2}{Y'_n} \int_{-2(\hat{K}_2/\hat{K}_1)Y_n'^{-1/2}}^{+2(\hat{K}_2/\hat{K}_1)Y_n'^{-1/2}} \int_{-2\hat{K}_1Y_n'^{-1/2}}^{+2\hat{K}_1Y_n'^{-1/2}} d\xi d\eta = \mathcal{O}(\hat{y}_n^{-4\beta})$$

and for $\hat{\mathbf{C}}_2''^{(i,n)}$:

$$\text{Leb}(\hat{\mathbf{C}}_2''^{(i,n)}) \leq \frac{1}{Y'_n} \int_{\frac{1}{2}(\hat{K}_2/\hat{K}_1)Y_n'^{-1/2}}^A \int_{-2\hat{K}_2/\eta Y_n'}^{+2\hat{K}_2/\eta Y_n'} d\xi d\eta = \mathcal{O}(\hat{y}_n^{-4\beta} \log \hat{y}_n).$$

Therefore we finally have:

$$\text{Leb}(\hat{\mathbf{C}}_2^{(i,n)}) \leq C_{\#} \hat{y}_n^{-4\beta} \log \hat{y}_n$$

and summing over i and n we obtain

$$\text{Leb}(\hat{\mathbf{C}}_2) < \infty \text{ if } \sum_n n^{-\frac{4\beta}{\gamma}} \log n < \infty,$$

where the series converges if $\beta > 1/2$, that is, $\gamma > 2$; to conclude, notice that, by the argument used to prove item (c₁), if $\gamma \leq 2$, then $\bar{\mathbf{C}}_2$ has infinite measure, which implies that the same is true for $\hat{\mathbf{C}}_2$ and concludes the proof of the lemma. \square

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